VINAYAKAR MATRICULATION HIGHER SECONDARY SCHOOL, SIVAKASI.

11th Standard – Maths-CHAPTER 7

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EXAMPLES OF CHAPTER 7

Example 7.1

Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

Solution

The number of elements is the product of number of rows and number of columns. Therefore, we will find all ordered pairs of natural numbers whose product is 12. Thus, all the possible orders of the matrix are $1\times 12,\ 12\times 1,\ 2\times 6,\ 6\times 2,\ 3\times 4$ and 4×3.

Since 7 is prime, the only possible orders of the matrix are 1×7 and 7×1.

Example 7.2

Construct a 2×3 matrix whose $(i, j)^{\text{th}}$ element is given by

$$a_{ij} = \frac{\sqrt{3}}{2}|2i-3j| \quad (1 \le i \le 2,\ 1 \le j \le 3).$$

Solution

In general, a 2×3 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

By definition of a_{ij}, we easily have $a_{11} = \frac{\sqrt{3}}{2}|2-3| = \frac{\sqrt{3}}{2}$ and other entries of the matrix A may be computed similarly. Thus, the required matrix A is $\begin{bmatrix} \frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2} \end{bmatrix}$.

Example 7.3

Find x, y, a, and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$.

Solution

As the orders of the two matrices are same, they are equal if and only if the corresponding entries are equal. Thus, by comparing the corresponding elements, we get

$3x+4y=2,\ x-2y=4,\ a+b=5,$ and $2a-b=-5$.

Solving these equations, we get $x=2,\ y=-1,\ a=0,$ and $b=5$.

Example 7.4

Compute $A + B$ and $A - B$ if

$$A=\begin{bmatrix}4 & \sqrt{5} & 7\\ -1 & 0 & 0.5\end{bmatrix} \text{ and } B=\begin{bmatrix}\sqrt{3} & \sqrt{5} & 7.3\\ 1 & \frac{1}{3} & \frac{1}{4}\end{bmatrix}.$$

Solution

By the definitions of addition and subtraction of matrices, we have

$$A+B=\begin{bmatrix}4+\sqrt{3} & 2\sqrt{5} & 14.3\\ 0 & \frac{1}{3} & \frac{3}{4}\end{bmatrix} \text{ and } A-B=\begin{bmatrix}4-\sqrt{3} & 0 & -0.3\\ -2 & -\frac{1}{3} & \frac{1}{4}\end{bmatrix}$$

Example 7.5

Find the sum $A + B + C$ if A, B, C are given by

$$A=\begin{bmatrix}\sin^2\theta & 1\\ \cot^2\theta & 0\end{bmatrix},\ B=\begin{bmatrix}\cos^2\theta & 0\\ -\operatorname{cosec}^2\theta & 1\end{bmatrix} \text{ and } C=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Solution

By the definition of sum of matrices, we have

$$A+B+C=\begin{bmatrix}\sin^2\theta+\cos^2\theta+0 & 1+0-1\\ \cot^2\theta-\operatorname{cosec}^2\theta-1 & 0+1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}.$$

Example 7.6

Determine $3B + 4C - D$ if B, C, and D are given by

$$B=\begin{bmatrix}2 & 3 & 0\\ 1 & -1 & 5\end{bmatrix},\ C=\begin{bmatrix}-1 & -2 & 3\\ -1 & 0 & 2\end{bmatrix},\ D=\begin{bmatrix}0 & 4 & -1\\ 5 & 6 & -5\end{bmatrix}.$$

Solution

$$3B+4C-D=\begin{bmatrix}6 & 9 & 0\\ 3 & -3 & 15\end{bmatrix}+\begin{bmatrix}-4 & -8 & 12\\ -4 & 0 & 8\end{bmatrix}+\begin{bmatrix}0 & -4 & 1\\ -5 & -6 & 5\end{bmatrix}=\begin{bmatrix}2 & -3 & 13\\ -6 & -9 & 28\end{bmatrix}.$$

Example 7.7

Simplify :

$$\sec\theta\begin{bmatrix}\sec\theta & \tan\theta\\ \tan\theta & \sec\theta\end{bmatrix}-\tan\theta\begin{bmatrix}\tan\theta & \sec\theta\\ \sec\theta & \tan\theta\end{bmatrix}.$$

Solution

If we denote the given expression by A, then using the scalar multiplication rule, we get

$$A=\begin{bmatrix}\sec^2\theta & \sec\theta\tan\theta\\ \sec\theta\tan\theta & \sec^2\theta\end{bmatrix}-\begin{bmatrix}\tan^2\theta & \tan\theta\sec\theta\\ \sec\theta\tan\theta & \tan^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Example 7.8

If $A=\begin{bmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{bmatrix}$, compute A^2.

Solution

$$A^2=AA=\begin{bmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{bmatrix}\begin{bmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix}$$

Example 7.9

Solve for x if $\begin{bmatrix}x & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ -1 & -4 & 1\\ -1 & -1 & -2\end{bmatrix}\begin{bmatrix}x\\ 2\\ 1\end{bmatrix}=O.$

Solution

$$\begin{bmatrix}x & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ -1 & -4 & 1\\ -1 & -1 & -2\end{bmatrix}\begin{bmatrix}x\\ 2\\ 1\end{bmatrix} = O$$

That is, $$\begin{bmatrix}x-2+1 & x-8+1 & 2x+2+2\end{bmatrix}\begin{bmatrix}x\\ 2\\ 1\end{bmatrix} = O$$

$$\begin{bmatrix}x-1 & x-7 & 2x+4\end{bmatrix}\begin{bmatrix}x\\ 2\\ 1\end{bmatrix} = O$$

$$x(x-1)+2(x-7)+1(2x+4) = 0$$

$$x^2+3x-10 = 0 \Rightarrow x=-5,\ 2.$$

Example 7.10

If $A=\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 0 & -3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & -3\\ -1 & 1\\ 1 & 2\end{bmatrix}$ find AB and BA if they exist.

Solution

The order of A is 3×3 and the order of B is 3×2. Therefore the order of AB is 3×2.

A and B are conformable for the product AB. Call $C = AB$. Then,

c_{11} = (first row of A) (first column of B)

$$\Rightarrow \quad c_{11} = [1 \quad -1 \quad 2]\begin{bmatrix}1\\ -1\\ 1\end{bmatrix} = 1+1+2 = 4, \quad \text{since } c_{11} \text{ is an element.}$$

Similarly $c_{12} = 0, c_{21} = 0, c_{22} = 13, c_{31} = 7, c_{32} = 5.$

Example 7.11

A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges and 3 pomegranates. Pack-II contains 5 apples, 4 oranges and 4 pomegranates and Pack –III contains 6 apples, 6 oranges and 6 pomegranates. The cost of an apple, an orange and a pomegranate respectively are ₹ 30, ₹ 15 and ₹ 45. What is the cost of preparing each package of fruits?

Solution

Cost matrix $A = [30 \quad 15 \quad 45]$, Fruit matrix

$$B = \begin{array}{c|c} \begin{array}{ccc} \text{P-I} & \text{P-II} & \text{P-III} \end{array} & \\ \begin{bmatrix}6 & 5 & 6\\ 3 & 4 & 6\\ 3 & 4 & 6\end{bmatrix} & \begin{array}{l}\text{Apples}\\ \text{Oranges}\\ \text{Pomegranates}\end{array}\end{array}$$

Cost of packages are obtained by computing AB. That is, by multiplying cost of each item in A (cost matrix A) with number of items in B (Fruit matrix B).

$$AB = [30 \quad 15 \quad 45]\begin{bmatrix}6 & 5 & 6\\ 3 & 4 & 6\\ 3 & 4 & 6\end{bmatrix} = \begin{bmatrix}360\\ 390\\ 540\end{bmatrix}$$

Pack-I cost ₹ 360, Pack-II cost ₹ 390, Pack-III costs ₹ 540.

Example 7.12

If $A=\begin{bmatrix}4 & 6 & 2\\0 & 1 & 5\\0 & 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1 & -1\\3 & -1 & 4\\-1 & 2 & 1\end{bmatrix}$,

verify (i) $(AB)^T=B^TA^T$ (ii) $(A+B)^T=A^T+B^T$ (iii) $(A-B)^T=A^T-B^T$ (iv) $(3A)^T=3A^T$

Solution

(i) $$AB=\begin{bmatrix}4 & 6 & 2\\0 & 1 & 5\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & -1\\3 & -1 & 4\\-1 & 2 & 1\end{bmatrix}=\begin{bmatrix}16 & 2 & 22\\-2 & 9 & 9\\7 & 1 & 14\end{bmatrix}$$

$$(AB)^T=\begin{bmatrix}16 & -2 & 7\\2 & 9 & 1\\22 & 9 & 14\end{bmatrix} \qquad \ldots (1)$$

$$B^T=\begin{bmatrix}0 & 3 & -1\\1 & -1 & 2\\-1 & 4 & 1\end{bmatrix},\quad A^T=\begin{bmatrix}4 & 0 & 0\\6 & 1 & 3\\2 & 5 & 2\end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

(ii) $$A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \ldots (3)$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \ldots (4)$$

From (3) and (4), $(A+B)^T = A^T + B^T$.

(iii) $$A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad \ldots (5)$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad \ldots (6)$$

From (5) and (6) $(A-B)^T = A^T - B^T$.

(iv) $$3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3 \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = 3(A^T) \;.$$

Example 7.13

Express the matrix $A=\begin{bmatrix}1&3&5\\-6&8&3\\-4&6&5\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

Solution

$$A=\begin{bmatrix}1&3&5\\-6&8&3\\-4&6&5\end{bmatrix}\Rightarrow\ A^T=\begin{bmatrix}1&-6&-4\\3&8&6\\5&3&5\end{bmatrix}$$

$$\text{Let }\ P=\frac{1}{2}(A+A^T)=\frac{1}{2}\begin{bmatrix}2&-3&1\\-3&16&9\\1&9&10\end{bmatrix}$$

$$\text{Now }\ P^T=\frac{1}{2}\begin{bmatrix}2&-3&1\\-3&16&9\\1&9&10\end{bmatrix}=P$$

Thus, $P=\frac{1}{2}(A+A^T)$ is a symmetric matrix.

$$\text{Let } Q=\frac{1}{2}(A-A^T)$$

$$=\frac{1}{2}\begin{bmatrix}0&9&9\\-9&0&-3\\-9&3&0\end{bmatrix}$$

$$\text{Then }\ Q^T=\frac{1}{2}\begin{bmatrix}0&-9&-9\\9&0&3\\9&-3&0\end{bmatrix}=-Q$$

Thus $Q=\frac{1}{2}(A-A^T)$ is a skew-symmetric matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix}2&-3&1\\-3&16&9\\1&9&10\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0&9&9\\-9&0&-3\\-9&3&0\end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

Example 7.14

Evaluate : (i) $\begin{vmatrix}2&4\\-1&2\end{vmatrix}$ (ii) $\begin{vmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{vmatrix}$.

Solution

(i) $\begin{vmatrix}2&4\\-1&2\end{vmatrix}=(2\times2)-(-1\times4)=4+4=8.$

(ii) $\begin{vmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{vmatrix}=(\cos\theta\cos\theta)-(-\sin\theta\sin\theta)=\cos^2\theta+\sin^2\theta=1.$

Example 7.15

Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$. Also check that $|A|$ remains unaltered by expanding along any row or any column.

Solution

Minors :

$$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10-30 = -40$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} = 8+18 = 26$$

$$M_{13} = \begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix} = 20-15 = 5$$

$$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6+10 = 16$$

$$M_{22} = \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = 2-6 = -4$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} = 5+9 = 14$$

$$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18-10 = 8$$

$$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6+8 = 14$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5-12 = -17$$

Cofactors :

$$A_{11} = (-1)^{1+1}(-40) = -40$$

$$A_{12} = (-1)^{1+2}(+26) = -26$$

$$A_{13} = (-1)^{1+3}(5) = 5$$

$$A_{21} = (-1)^{2+1}(16) = -16$$

$$A_{22} = (-1)^{2+2}(-4) = -4$$

$$A_{23} = (-1)^{2+3}(14) = -14$$

$$A_{31} = (-1)^{3+1}(8) = 8$$

$$A_{32} = (-1)^{3+2}(14) = -14$$

$$A_{33} = (-1)^{3+3}(-17) = -17$$

Expanding along R_1 yields

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ .$$

$$|A| = 1(-40) + (3)(-26) + (-2)(5) = -128 \ . \qquad \ldots (3)$$

Expanding along C_1 yields

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \ .$$

$$= 1(-40) + 4(-16) + -3(8) = -128 \qquad \ldots (4)$$

From (3) and (4), we have

|A| obtained by expanding along R_1 is equal to expanding along C_1.

Example 7.16

Find $|A|$ if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{bmatrix}$.

Solution

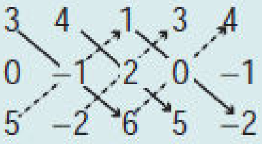
$$\begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0M_{11} - \sin\alpha M_{12} + \cos\alpha M_{13}$$

$$= 0 - \sin\alpha(0 - \cos\alpha\sin\beta) + \cos\alpha(-\sin\alpha\sin\beta - 0) = 0.$$

Example 7.17

Compute $|A|$ using Sarrus rule if $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$.

Solution



$$|A| = [3(-1)(6) + 4(2)(5) + 1(0)(-2)] \ -[5(-1)(1) + (-2)(2)3 + 6(0)(4)]$$

$$= [-18 + 40 + 0] - [-5 - 12 + 0] \ = 22{+}17 = 39.$$

Example 7.18

If a, b, c and x are positive real numbers, then show that $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ is zero.

Solution

Applying $C_1 \to C_1 - C_2$, we get $\begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$, since C_1 and C_3 are proportional.

Example 7.19

Without expanding the determinants, show that $|B| = 2|A|$.

Where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Solution

We have
$$|B| = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2(-1)^2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2|A|.$$

Example 7.20

Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 2017-2014 & 0 \\ 2020 & 2023-2020 & 1 \\ 2023 & 2026-2023 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix}$$

$$= -3(2014 - 2023) = -3(-9) = 27.$$

Example 7.21

Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$.

Solution

Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3) = 0$ which gives $x = 1, 2, 3$.

Example 7.22

Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

Solution

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\text{LHS} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)[(z+x)-(y+x)] \;.$$

$$= (y-x)(z-x)(z-y) \;.$$

$$= (x-y)(y-z)(z-x) = \text{RHS}.$$

Example 7.23

Using Factor Theorem, prove that $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$.

Solution

$$\text{Let } |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} .$$

$$\text{Putting } x = 1, \text{ we get } |A| = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

Since all the three rows are identical, $(x-1)^2$ is a factor of $|A|$

$$\text{Putting } x = -9 \text{ in } |A| \text{,we get} |A| = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0$$

Therefore $(x+9)$ is a factor of $|A|$ [since $C_1 \to C_1 + C_2 + C_3$].

The product $(x-1)^2(x+9)$ is a factor of $|A|$. Now the determinant is a cubic polynomial in x.

Therefore the remaining factor must be a constant 'k'.

$$\text{Therefore } \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x-1)^2(x+9).$$

Equating x^3 term on both sides, we get $k = 1$. Thus $|A| = (x-1)^2(x+9)$.

Example 7.24

Prove that $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)\ (y-z)\ (z-x)\ (xy+yz+zx)$.

Solution

Let $$|A| = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$$

Putting $x = y$ gives $$|A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{since } R_1 \equiv R_2).$$

Therefore $(x-y)$ is a factor.

The given determinant is in cyclic symmetric form in x, y and z. Therefore $(y-z)$ and $(z-x)$ are also factors.

The degree of the product of the factors $(x-y)(y-z)(z-x)$ is 3 and the degree of the product of the leading diagonal elements $1 \times y^2 \times z^3$ is 5.

Therefore the other factor is $k(x^2+y^2+z^2)+\ell(xy+yz+zx)$.

Thus $$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = [k(x^2+y^2+z^2)+\ell(xy+yz+zx)]\times(x-y)(y-z)(z-x).$$

Putting $x=0$, $y=1$ and $z=2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = [k(0+1+4)+\ell(0+2+0)](-1)(1-2)(2-0)$$

$$\Rightarrow \quad (8-4) = [(5k+2\ell)](-1)(-1)(2)$$

$$4 = 10k+4\ell \Rightarrow 5k+2\ell = 2. \qquad \text{... (1)}$$

Putting $x=0, y=-1$ and $z=1$, We get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2)+\ell(-1)](1)(-2)(1)$$

$$\Rightarrow \quad [(2k-\ell)(-2)] = 2$$

$$2k-\ell = -1. \qquad \text{... (2)}$$

Solving (1) and (2), we get $k=0, \ell=1$.

Therefore $$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$$

Example 7.25

Prove that $|A| = \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pqr\ (p+q+r)^3.$

Solution :

Taking $p=0$, we get $|A| = \begin{vmatrix} (q+r)^2 & 0 & 0 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix} = 0.$

Therefore, $(p-0)$ is a factor. That is, p is a factor.

Since $|A|$ is in cyclic symmetric form in p, q, r and hence q and r also factors.

Putting $p+q+r=0 \Rightarrow q+r=-p\,;\, r+p=-q\ ;$ and $p+q=-r$.

$$|A| = \begin{vmatrix} p^2 & p^2 & p^2 \\ q^2 & q^2 & q^2 \\ r^2 & r^2 & r^2 \end{vmatrix} = 0 \text{ since 3 columns are identical.}$$

Therefore, $(p+q+r)^2$ is a factor of $|A|$.

The degree of the obtained factor $pqr\ (p+q+r)^2$ is 5. The degree of $|A|$ is 6.

Therefore, required factor is $k\ (p+q+r)$.

$$\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k(p+q+r)\ (p+q+r)^2 \times pqr$$

Taking $p=1,\ q=1,\ c=1$, we get

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1+1+1)^3\ (1)\ (1)\ (1)\,.$$

$4(16-1)-1(4-1)+1(1-4)=27k$

$60-3-3=27\,k \Rightarrow k=2.$

$|A| = 2pqr\ (p+q+r)^3.$

Example 7.26

In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0,$
prove that ΔABC is an isosceles triangle.

Solution :

By putting $\sin A = \sin B$, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin A & 1+\sin C \\ \sin A(1+\sin A) & \sin A(1+\sin A) & \sin C(1+\sin C) \end{vmatrix} = 0$$

That is, by putting $\sin A = \sin B$ we see that, the given equation is satisfied.
Similarly by putting $\sin B = \sin C$ and $\sin C = \sin A$, the given equation is satisfied.
Thus, we have $A=B$ or $B=C$ or $C=A$.
In all cases atleast two angles are equal. Thus the triangle is isosceles.

Example 7.27

Verify that $|AB| = |A|\ |B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Solution

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |AB| = 1. \qquad \text{... (1)}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1.$$

$$|B| = \cos^2\theta + \sin^2\theta = 1.$$

$$|A|\ |B| = 1. \qquad \text{... (2)}$$

From (1) and (2), $|AB| = |A|\ |B|$.

Example 7.28

Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ab & bc & a^2+b^2 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}.$$

$$= \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix} = \text{RHS}.$$

Example 7.29

Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$.

Solution

$$\text{RHS} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \quad [\text{In the } 2^{\text{nd}} \text{ determinant } R_2 \leftrightarrow R_3]$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Taking row by column method, we get

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{RHS.}$$

Example 7.30

Prove that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times (-1)\,(-1) \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}.$$

$$= \text{R.H.S.}$$

Example 7.31

If A_i, B_i, C_i are the cofactors of a_i, b_i, c_i, respectively, $i = 1$ to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2 .$$

Example 7.31

If A_i, B_i, C_i are the cofactors of a_i, b_i, c_i, respectively, $i = 1$ to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2.$$

Solution

Consider the product $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

That is, $|A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$.

$$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2.$$

Example 7.32

If the area of the triangle with vertices (– 3, 0), (3, 0) and (0, k) is 9 square units, find the values of k.

Solution

$$\text{Area of the triangle} = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$9 = \left| \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} \right| = \left| \frac{1}{2}(-k)(-3-3) \right|$$

$$\Rightarrow \quad 9 = 3|k| \text{ and hence, } k = \pm 3.$$

Example 7.33

Find the area of the triangle whose vertices are (– 2, – 3), (3, 2), and (– 1, – 8).

Solution

$$\text{Area of the triangle} = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

$$\left| \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \right| = \left| \frac{1}{2}(-20+12-22) \right| = |-15| = 15$$

and therefore required area is 15 sq.units.

Example 7.34

Show that the points $(a, b + c)$, $(b, c + a)$, and $(c, a + b)$ are collinear.

Solution

To prove the given points are collinear, it suffices to prove $|A| = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = 0$.

Applying $C_1 \to C_1 + C_2$, we deduce that

$$|A| = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = (a+b+c) \times 0 = 0$$

which shows that the given points are collinear.