DIFFERENTIAL CALCULUS

Differentiation: (5 Marks). one may expect 1-24 one Question (or) 25-31 one. or both from 1-24 (X)

1. S.T the derivative x^n is nx^{n-1} from the first Principle.

(ie) $\frac{d}{dx}(x^n) = nx^{n-1}$

Note

$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Proof: $y = f(x) = x^n$

$y+\Delta y \; f(x+\Delta x) = (x+\Delta x)^n \Rightarrow \Delta y = (x+\Delta x)^n - x^n$

Now $\frac{d}{dx}(f(x)) = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{x^n\left[\left(1+\frac{\Delta x}{x}\right)^n - 1\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} x^{n-1}\frac{\left[\left(1+\frac{\Delta x}{x}\right)^n - 1\right]}{\frac{\Delta x}{x}}.$$

Let $1+\frac{\Delta x}{x} = y$

as $\Delta x \to 0$ $y \to 1$

$$\therefore \frac{d}{dx}(x^n) = x^{n-1} \lim_{y \to 1} \frac{y^n - 1^n}{y-1}$$

$$= x^{n-1} . n \, 1^{n-1}$$

$$= n \cdot x^{n-1}$$

2) S.T $\frac{d}{dx}(\sin x) = \cos x$ by First principle.

Sol: Let $y = f(x) = \sin x$

$y+\Delta y = f(x+\Delta x) = \sin(x+\Delta x)$

$\Delta y = \sin(x+\Delta x) - \sin x$.

$= 2\cos\frac{(x+\Delta x+x)}{2}\sin\left(\frac{\cancel{x}+\Delta x-\cancel{x}}{2}\right)$

$\frac{\Delta y}{\Delta x} = \frac{2\cos\frac{(2x+\Delta x)}{2}\sin\frac{\Delta x}{2}}{\Delta x}$

$= \frac{2\cos\left(x+\frac{\Delta x}{2}\right)\cdot\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin\left(\frac{\Delta x}{2}\right)\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{2}\cdot\sin\frac{\Delta x}{2}}{\cancel{2}\,\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \cos\left(x+\frac{\Delta x}{2}\right)$$

$$= 1\cdot\cos x$$

$$= \cos x.$$

3) Find $\frac{dy}{dx}$ of $y = \cos x$ from the First principle.

Sol: $y = \cos x$

$$y+\Delta y = \cos(x+\Delta x)$$

$$\Delta y = \cos(x+\Delta x) - \cos x$$

$$= -2\sin\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\cancel{x}+\Delta x-\cancel{x}}{2}\right)$$

$$= -2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= -\frac{\cancel{2}}{\cancel{2}}\,\frac{\sin\left(x+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}}{\Delta x/2}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} -\sin\left(x+\frac{\Delta x}{2}\right) \cdot \lim_{\frac{\Delta x}{2} \to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\Delta x/2}\right)$$

$$= -\sin x \cdot 1$$

$$= -\sin x.$$

4. Find the derivative of $y = e^{7x}$ from the first principle.

Sol: Let $y = e^{7x}$

$$y+\Delta y = e^{7(x+\Delta x)}$$

$$= e^{7x}\cdot e^{7\Delta x}$$

$$\Delta y = e^{7x}\cdot e^{7\Delta x} - e^{7x}$$

$$= e^{7x}\left(e^{7\Delta x}-1\right) \Rightarrow \frac{\Delta y}{\Delta x} = 7\cdot e^{7x}\frac{\left(e^{7\Delta x}-1\right)}{7\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 7e^{7x} \frac{(e^{7\Delta x}-1)}{7\Delta x}$$ as $\Delta x \to 0, 7\Delta x \to 0$

$$\therefore 7e^{7x} \cdot \lim_{7\Delta x \to 0} \left(\frac{e^{7\Delta x}-1}{7\Delta x}\right)$$

$$= 7e^{7x}$$

5. Find the derivative of $\frac{x^2 + e^x \sin x}{\cos x + \log x}$ w.r.t. x.

Soul: Let $y = \frac{x^2 + e^x \sin x}{\cos x + \log x} = \frac{u}{v}$

$$u' = 2x + (e^x \cos x + e^x \sin x)$$

$$= 2x + e^x(\cos x + \sin x)$$

$$v' = -\sin x + \frac{1}{x}$$

$$= \frac{1 - x\sin x}{x}$$

$$\frac{dy}{dx} = \frac{\cancel{uv' - vu'}}{\cancel{v^2}} = \frac{\cancel{x^2 e^x \sin x \left(\frac{1-x\sin x}{x}\right) - (\cos x + \log x)(2x + e^x(\cos x + \sin x)}}{\cancel{(\cos x + \log x)^2}}$$

$$= x^2 +$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(\cos x + \log x)(2x + e^x(\sin x + \cos x)) - (x^2 + e^x \sin x)\left(\frac{1}{x} - \sin x\right)}{(\cos x + \log x)^2}$$

6) Find the derivative of $\frac{\sin x + x\cos x}{x\sin x - \cos x}$ w.r.t. x.

Sol: Let $y = \frac{\sin x + x\cos x}{x\sin x - \cos x} = \frac{u}{v}$

$$u' = \cos x + (-x\sin x + \cos x) = 2\cos x - x\sin x$$

$$v' = x\cos x + \sin x + \sin x = 2\sin x + x\cos x.$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x\sin x - \cos x)(2\cos x - x\sin x) - (\sin x + x\cos x)(x\cos x + 2\sin x)}{(x\sin x - \cos x)^2}$$

$= (2x \cancel{\sin x} \cos x - x^2 \sin^2 x) - (x \cancel{\sin x} \cos x + 2\sin^2 x + x^2\cos^2 x$
$- 2\cos^2 x + x \cancel{\sin x} \cos x$ $+ 2x \cancel{\sin x} \cos x)$

$$= \frac{-x^2\sin^2 x - x^2\cos^2 x - 2\sin^2 x - 2\cos^2 x}{(x\sin x - \cos x)^2}$$

$$= \frac{-[x^2(\sin^2 x + \cos^2 x) + 2(\sin^2 x + \cos^2 x)]}{(x\sin x - \cos x)^2}$$

$$= \frac{-(x^2+2)}{(x\sin x - \cos x)^2}$$

Inverse function:

7) Find the derivative of $y = \sin^{-1}(x^2+2x)$

Sol: Let $u = x^2 + 2x$

$$\frac{du}{dx} = 2x+2 = 2(x+1)$$

$\therefore$ $y = \sin^{-1}(u)$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-(x^2+2x)^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1-(x^2+2x)^2}} \times 2(x+1)$$

8) Find the derivative of $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

Sol: $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

$$\frac{dy}{dx} = \frac{1(-\operatorname{cosec}^2 x)}{1+\cot^2 x} + \frac{-1}{1+\tan^2 x}\cdot(\sec^2 x)$$

$$= \frac{-\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} - \frac{1}{\sec^2 x}\cdot\sec^2 x$$

$$= -1 - 1$$

$$= -2$$

9) Differentiate $\dfrac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}$

Note: In the given problem there are 3 or more functions and also the variable is in power then by taking log on both sides and then differentiate.

Sol: $y = \dfrac{(1-x)(\sqrt{x^2+2})}{(x+3)(\sqrt{x-1})}$

$$\log y = \log(1-x) + \log(x^2+2)^{1/2} - \log(x+3) - \log(x-1)^{1/2}$$

$$= \log(1-x) + \frac{1}{2}\log(x^2+2) - \log(x-3) - \frac{1}{2}\log(x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{1-x} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+2} - \frac{1}{x-3} - \frac{1}{2}\cdot\frac{1}{x-1}$$

$$= \frac{x}{x^2+2} - \frac{1}{x-3} + \frac{1}{x-1} + \frac{1}{2}\frac{1}{x-1}$$

$$= \frac{x}{x^2+2} - \frac{1}{x-3} + \frac{1}{x-1} \Rightarrow \frac{dy}{dx} = y\left[\frac{-x}{x^2+2} - \frac{1}{x-3} + \frac{1}{x-1}\right]$$

$$= \frac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}\left(\frac{x}{x^2+2} - \frac{1}{x-3} + \frac{1}{x-1}\right)$$

10. If $y = \dfrac{(x^2+2)(x+\sqrt{2})}{\sqrt{x+4}\,(x-7)}$ find $\dfrac{dy}{dx}$.

Sol: $y = \dfrac{(x^2+2)(x+\sqrt{2})}{\sqrt{x+4}\,(x-7)}$

$$\log y = \log(x^2+2) + \log(x+\sqrt{2}) - \log\sqrt{x+4} - \log(x-7)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1\times 2x}{x^2+2} + \frac{1}{x+\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{x+4} - \frac{1}{x-7}$$

$$\frac{dy}{dx} = y\left[\frac{2x}{x^2+2} + \frac{1}{x+\sqrt{2}} - \frac{1}{2(x+4)} - \frac{1}{x-7}\right]$$

$$= \frac{(x^2+2)(x+\sqrt{2})}{\sqrt{x+4}\,(x-7)}\left[\frac{2x}{x^2+2} + \frac{1}{x+\sqrt{2}} - \frac{1}{2(x+4} - \frac{1}{x-7}\right]$$

11) If $y = \dfrac{\sin x \cos(e^x)}{e^x + \log x}$ find $\dfrac{dy}{dx}$.

Sol: $y = \dfrac{\sin x \cdot \cos(e^x)}{e^x + \log x}$

$$\log y = \log \sin x + \log \cos(e^x) - \log(e^x - \log(\log x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{-\sin e^x \cdot e^x}{\cos e^x} - \frac{1\left(e^x - \frac{1}{x}\right)}{e^x - \log x}$$

$$= \tan x - e^x \tan e^x - \frac{\left(e^x - \frac{1}{x}\right)}{e^x - \log x}.$$

$$\frac{dy}{dx} = y\left[\tan x - e^x \tan e^x - \frac{\left(e^x - \frac{1}{x}\right)}{e^x - \log x}\right]$$

$$= \frac{\sin x \cos(e^x)}{e^x + \log x}\left[\tan x - e^x \tan e^x - \left(\frac{e^x - \frac{1}{x}}{e^x - \log x}\right)\right]$$

Method of Substitution

12) If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ find $\frac{dy}{dx}$.

Sol: $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan t \Rightarrow t = \tan^{-1}x$.

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sqrt{1+\tan^2 t}-1}{\tan t} = \frac{\sec t - 1}{\tan t} = \frac{\frac{1-\cos t}{\cos t}}{\frac{\sin t}{\cos t}}$$

$$= \frac{\cancel{2}\sin^{\cancel{2}} t/2}{\cancel{2}\cancel{\sin t/2}\cos t/2}$$

$$= \tan t/2$$

$\therefore\ y = \tan^{-1}(\tan t/2)$

$= t/2$

$y = \frac{1}{2}.\tan^{-1}x.$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2}$$

13) If $y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ find $\frac{dy}{dx}$.

Sol: $y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put $x = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x.$

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}$$

$$= \frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}$$

$$= \frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}$$

$$= + \frac{\cancel{\sqrt{2}}(\cos\theta - \sin\theta)}{\cancel{\sqrt{2}}(\cos\theta + \sin\theta)}$$

$\div \cos\theta$ $$\frac{1-\tan\theta}{1+\tan\theta} = \frac{\tan(\pi/4) - \tan\theta}{\tan \pi/4 + \tan\theta}$$

$$= \tan\left(\frac{\pi}{4} - \theta\right)$$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right) = \tan^{-1}\left(\tan(\pi/4 - \theta)\right)$$

$$y = \frac{\pi}{4} - \theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x.$$

$$\frac{dy}{dx} = 0 - \frac{1}{2}\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{2\sqrt{1-x^2}}$$

14) If $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$

$$\sqrt{1+\sin x} = \sqrt{\sin^2 x/2 + \cos^2 x/2 + 2\sin x/2 \cos x/2}$$

$$= \sqrt{(\sin x/2 + \cos x/2)^2}$$

$$= \sin x/2 + \cos x/2$$

$$\sqrt{1-\sin x} = \sqrt{\sin^2 x/2 + \cos^2 x/2 - 2\sin x/2 \cos x/2}$$

$$= \sqrt{(\sin x/2 - \cos x/2)^2}$$

$$= \sin x/2 - \cos x/2$$

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \frac{\cancel{2}\sin x/2}{\cancel{2}\cos x/2}$$

$$= \tan x/2$$

$$= \cot\left(\frac{\pi}{2} - x/2\right)$$

$\therefore \quad y = \cot^{-1}\left(\cot\left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$

$= \frac{\pi}{2} - \frac{x}{2}$

$\frac{dy}{dx} = -\frac{1}{2}$

15) Parametric functions.

If $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$ find $\frac{dy}{dx}$.

Sol: $x = 2\cos\theta - \cos 2\theta$

$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$.

$= 2(\sin 2\theta - \sin\theta)$

$= 2 \cdot 2\cos\frac{3\theta}{2}\sin\theta/2$

$y = 2\sin\theta - \sin 2\theta$

$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$

$= -2(\cos 2\theta - \cos\theta)$

$= -2 \cdot (-2\sin 3\theta/2 \sin\theta/2)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{4}\sin 3\theta/2 \,\cancel{\sin\theta/2}}{\cancel{4}\cos 3\theta/2 \,\cancel{\sin\theta/2}}$

$= \tan 3\theta/2$

16. $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

Sol: $x = \frac{3at}{1+t^3}$

$\frac{dx}{dt} = \frac{(1+t^3)\cdot 3a - 3at\cdot 3t^2}{(1+t^3)^2}$

$= \frac{3a + 3at^3 - 9at^3}{(1+t^3)^2}$

$= \frac{3a - 6at^3}{(1+t^3)^2}$

$= \frac{3a(1-2t^3)}{(1+t^3)^2}$

$y = \frac{3at^2}{1+t^3}$

$\frac{dy}{dt} = \frac{(1+t^3)6at - 3at^2\cdot 3t^2}{(1+t^3)^2}$

$= \frac{6at + 6at^4 - 9at^4}{(1+t^3)^2}$

$= \frac{6at - 3at^4}{(1+t^3)^2} = \frac{3at(2-t^3)}{(1+t^3)^2}$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3at(2-t^2)}{\cancel{(1+t^3)^2}} \times \frac{\cancel{(1+t^3)^2}}{3a(1-2t^3)}$

$= \frac{t(2-t^2)}{1-2t^3}$

17) If $x = a(\cos\theta + \log\tan\theta/2)$ $y = a\sin\theta$ find dy/dx.

Sol: $x = a(\cos\theta + \log\tan\theta/_2)$

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\theta/_2}\cdot\sec^2\theta/_2\cdot\frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{\cos\theta/_2}{\sin\theta/_2}\cdot\frac{1}{\cos^2\theta/_2}\cdot\frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(\frac{1-\sin^2\theta}{\sin\theta}\right)$$

$$= a\left(\frac{\cos^2\theta}{\sin\theta}\right)$$

$y = a\sin\theta$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\cos\theta.\sin\theta}{\cancel{a}\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Implicit functions

18) Find $\frac{dy}{dx}$ when $\tan(x+y) + \tan(x-1) = 1$

Sol: $\tan(x+y) + \tan(x-y) = 1$

$$\sec^2(x+y)\left(1+\frac{dy}{dx}\right) + \sec^2(x-y)\left(1-\frac{dy}{dx}\right) = 0$$

$$\sec^2(x+y) + \sec^2(x-y) = \sec^2(x-y)\frac{dy}{dx} - \sec^2(x+y)\frac{dy}{dx}$$

$$= \frac{dy}{dx}\left(\sec^2(x-y) - \sec^2(x+y)\right)$$

$$\frac{\sec^2(x+y) + \sec^2(x-y)}{\sec^2(x-y) - \sec^2(x+y)} = \frac{dy}{dx}$$

19) Find $\frac{dy}{dx}$ if $xy + e^{-y}x + ye^{x} = x^2$

Sol: $xy + xe^{-y} + ye^{x} = x^2$

$$x\frac{dy}{dx} + y + xe^{-y}(-1)\frac{dy}{dx} + e^{-y} + ye^{x} + e^{x}\frac{dy}{dx} = 2x.$$

$$x\frac{dy}{dx} - xe^{-y}\frac{dy}{dx} + e^{x}\frac{dy}{dx} = 2x - y - e^{-y} - ye^{x}$$

$$\frac{dy}{dx}(x - xe^{-y} + e^{x}) = 2x - y - e^{-y} - ye^{x}$$

$$\frac{dy}{dx} = \frac{2x - y - e^{-y} - ye^{x}}{x - xe^{-y} + e^{x}}$$

19) Find $\frac{dy}{dx}$ if $e^{x} + e^{y} = e^{x+y}$

Sol: $e^{x} + e^{y} = e^{x+y}$

$$e^{x} + e^{y}\cdot\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$e^{x} + e^{y}\frac{dy}{dx} = e^{x+y} + e^{x+y}\cdot\frac{dy}{dx}$$

$$e^{y}\frac{dy}{dx} - e^{x+y}\cdot\frac{dy}{dx} = e^{x+y} - e^{x}$$

$$\frac{dy}{dx}(e^{y} - e^{x+y}) = e^{x+y} - e^{x}$$

$$\frac{dy}{dx} = \frac{e^{x}\cdot e^{y} - e^{x}}{e^{y} - e^{x}\cdot e^{y}}$$

$$\frac{dy}{dx} = \frac{e^{x}(e^{y} - 1)}{e^{y}(1 - e^{x})} = e^{x-y}\left(\frac{e^{y} - 1}{1 - e^{x}}\right)$$

20) Find $\frac{dy}{dx}$ if $x^{y} = y^{x}$.

Sol: $x^{y} = y^{x}$

$$\log x^{y} = \log y^{x}$$

$$y\log x = x\log y.$$

$$y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx} = x\cdot\frac{1}{y}\frac{dy}{dx} + \log y$$

$$\frac{y}{x} - \log y = \frac{x}{y}\frac{dy}{dx} - \log x\frac{dy}{dx}$$

$$= \frac{dy}{dx}\left(\frac{x}{y} - \log x\right)$$

$$\frac{\frac{y - x\log y}{x}}{\frac{x - y\log x}{y}} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\frac{y - x\log y}{x - y\log x}\right]$$

22. Find $\frac{dy}{dx}$ if $x^m y^n = (x+y)^{m+n}$.

Sol: $x^m y^n = (x+y)^{m+n}$

$$m\log x + n\log y = (m+n)\log(x+y)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$$

$$\frac{n}{y}\frac{dy}{dx} - \frac{m+n}{x+y}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}.$$

$$\frac{dy}{dx}\left[\frac{n}{y} - \frac{m+n}{x+y}\right] = \frac{x(m+n) - m(x+y)}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{n(x+y) - (m+n)y}{y(x+y)}\right] = \frac{\cancel{mx} + nx - \cancel{mx} - my}{x(x+y)}$$

$$\frac{dy}{dx} = \frac{nx - my}{x\cancel{(x+y)}} \times \frac{y\cancel{(x+y)}}{nx + \cancel{ny} - my - \cancel{ny}}$$

$$= \frac{y}{x}\left(\frac{nx-my}{nx-my}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}$$

23) If $ax^2+by^2+2gx+2fy+2hxy+c=0$ S.T

$$\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$$

Sol: $ax^2+by^2+2gx+2fy+2hxy+c=0$

$$\cancel{2}ax + \cancel{2}by\frac{dy}{dx} + \cancel{2}g + \cancel{2}f\frac{dy}{dx} + \cancel{2}h\left(x\frac{dy}{dx}+y\right) = 0$$

$$by\frac{dy}{dx} + hx\frac{dy}{dx} + f\frac{dy}{dx} = -ax - g - hy$$

$$\frac{dy}{dx}(by + hx + f) = -(ax + hy + g)$$

$$\frac{dy}{dx} = -\frac{(ax+hy+g)}{by+hx+f}.$$

24) Find $\frac{dy}{dx}$ if $xy = 100(x+y)$

Sol: $xy = 100(x+y)$

$$x\frac{dy}{dx} + y = 100\left(1+\frac{dy}{dx}\right)$$

$$= 100 + 100\frac{dy}{dx}$$

$$x\frac{dy}{dx} - 100\frac{dy}{dx} = 100 - y$$

$$\frac{dy}{dx}(x-100) = 100 - y$$

$$\frac{dy}{dx} = \frac{100-y}{x-100}$$

Higher order derivatives

25) If $y = \log(\cos x)$ find y.

Sol: $y = \log \cos x$

$$\frac{dy}{dx} = \frac{-1 \sin x}{\cos x}$$

$$= -\tan x$$

$$\frac{d^2y}{dx^2} = -\sec^2 x$$

$$\frac{d^3y}{dx^3} = -2\sec x \cdot \sec x \tan x$$

$$= -2\sec^2 x \tan x.$$

26) If $y = e^{ax}\sin bx$ P.T $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$

Sol: $y = e^{ax}\sin bx$

$$\frac{dy}{dx} = be^{ax}\cos bx + a\sin bx\, e^{ax} = be^{ax}\cos bx + ay$$

$$\frac{d^2y}{dx^2} = b\left(-be^{ax}\sin bx + a\cos bx\, e^{ax}\right) + a\frac{dy}{dx}.$$

$$= -b^2y + a\left(\frac{dy}{dx} - ay\right) + a\frac{dy}{dx}$$

$\because a\cos bx\, e^{ax} = a\left(\frac{dy}{dx} - ay\right)$

$$= -b^2y + 2a\frac{dy}{dx} - a^2y$$

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0.$$

27) If $y = \cos(m \sin^{-1} x)$ P.T $(1-x^2)y_3 - 3xy_2 + (m^2-1)y_1 = 0$

Sol: $y = \cos(m \sin^{-1} x)$

$$y_1 = -\sin(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$y_1^2 = \frac{m^2 \sin^2(m \sin^{-1} x)}{(1-x^2)}$$

$$y_1^2(1-x^2) = m^2 \sin^2(m \sin^{-1} x)$$

$$= m^2(1-\cos^2(m \sin^{-1} x)) \quad (*)$$

$$= m^2(1-y^2)$$

$$y_1^2(-2x) + (1-x^2)2y_1 \cdot y_2 = m^2(-2yy_1)$$

$\div 2y_1$ $\quad -\cancel{2}y_1 x + (1-x^2)\cancel{2}y_2 = -\cancel{2}m^2 y$

$$-xy_2 - y_1 + (1-x^2)y_3 + y_2(-2x) = -m^2 \cdot y_1$$

$$(1-x^2)y_3 - 3xy_2 + (m^2-1)y_1 = 0.$$

28) If $y = e^{\tan^{-1} x}$ P.T $(1+x^2)y_2 + (2x-1)y_1 = 0$

Sol: $y = e^{\tan^{-1} x}$

$$y_1 = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$(1+x^2)y_1 = e^{\tan^{-1} x}$$

$$= y.$$

$$(1+x^2)y_2 + y_1 \cdot 2x = y_1$$

$$(1+x^2) + (2x-1)y_1 = 0.$$

29) If $y = x^3 - 1$ P.T $x^2 y_3 - 2xy_2 + 2y_1 = 0$

Sol: $y = x^3 - 1$

$y_1 = 3x^2$

$y_2 = 6x$

$y_3 = 6$

$\therefore$ LHS

$$x^2 y_3 - 2xy_2 + 2y_1 = 6x^2 - 2x(6x) + 2(3x^2)$$

$$= 6x^2 - 12x^2 + 6x^2$$

$$= 0.$$

30) If $x = \sin t$, $y = \sin pt$ P.T $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Sol: $x = \sin t$ $y = \sin pt$

$\frac{dx}{dt} = \cos t$ $\frac{dy}{dt} = p\cos pt$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p\cos pt}{\cos t}$

$\frac{dy}{dx} = p\frac{\sqrt{1-\sin^2 pt}}{\sqrt{1-\sin^2 t}}$.

$y_1 = p\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$y_1^2 = p^2\frac{(1-y^2)}{(1-x^2)}$ $)\cdot\frac{dt}{dx}$.

$(1-x^2)y_1^2 = p^2(1-y^2)$

$(1-x^2)2y_1y_2 + y_1^2(-2x) = p^2(0-2yy_1)$

$\div 2y_1$

$(1-x^2)y_1 - \cancel{2}xy_1 + p^2y = 0$.

31) If $x = a(\cos\theta + \theta\sin\theta)$ $y = a(\sin\theta - \theta\cos\theta)$

P.T $a\theta\frac{d^2y}{dx^2} = \sec^3\theta$.

$x = a(\cos\theta + \theta\sin\theta)$

$\frac{dx}{d\theta} = a(-\cancel{\sin\theta} + \theta\cos\theta + \cancel{\sin\theta})$

$= \theta\cos\theta$

$y = a(\sin\theta - \theta\cos\theta)$

$y = a(\cancel{\cos\theta} - (-\theta\sin\theta + \cancel{\cos\theta}))$

$= +a\theta\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cancel{\theta}\sin\theta}{\cancel{\theta}\cos\theta} \Rightarrow a\tan\theta$

$d^2y/dx^2 = a\sec^2\theta\frac{d\theta}{dx} = a\sec^2\theta\cdot\frac{1}{\cos\theta} = a\sec^3\theta$.

32) If $y = (\tan^{-1}x)^{\log x} + (\log x)^{\sin^{-1}x}$

Sol: $y = (\tan^{-1}x)^{\log x} + (\log x)^{\sin^{-1}x}$.

Let $u = (\tan^{-1}x)^{\log x}$, $v = (\log x)^{\sin^{-1}x}$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

$\log u = \log x(\log \tan^{-1}x)$

$\frac{1}{u}\frac{du}{dx} = \frac{\log x}{\log \tan^{-1}x} \times \frac{1}{1+x^2} + \frac{\log \tan^{-1}x}{x}$

$= \frac{\log x}{\log(\tan^{-1}x)(1+x^2)} + \frac{\log \tan^{-1}x}{x}$

$\frac{du}{dx} = u\left[\frac{\log x}{(1+x^2)\log \tan^{-1}x} + \frac{\log \tan^{-1}x}{x}\right]$

$= (\tan^{-1}x)^{\log x}\left[\frac{\log x}{(1+x^2)\log \tan^{-1}x} + \frac{\log \tan^{-1}x}{x}\right]$

$v = (\log x)^{\sin^{-1}x}$

$\log v = \sin^{-1}x \log(\log x)$

$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{\sin^{-1}x}{\log x}\cdot\frac{1}{x} + \frac{\log(\log x)}{\sqrt{1-x^2}}$

$\frac{dv}{dx} = v\left[\frac{\sin^{-1}x}{x\log x} + \frac{\log(\log x)}{\sqrt{1-x^2}}\right]$

$= (\log x)^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x\log x} + \frac{\log(\log x)}{\sqrt{1-x^2}}\right]$

$\therefore \frac{dy}{dx} = du/dx + dv/dx$

$= (\tan^{-1}x)^{\log x}\left[\frac{\log x}{(1+x^2)\log \tan^{-1}x} + \frac{\log \tan^{-1}x}{x}\right] + (\log x)^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x\log x} + \frac{\log(\log x)}{\sqrt{1-x^2}}\right]$

33) If $y = x^{\tan x} + \sin x^{\sin x}$

Sol: $y = x^{\tan x} + \sin x^{\sin x}$.

$u + v$

$u = x^{\tan x}$

$\log u = \tan x \log x$

$\frac{1}{u}\cdot\frac{du}{dx} = \frac{\tan x}{x} + \log x \cdot \sec^2 x$

$\frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x \sec^2 x\right]$

$= x^{\tan x}\left[\frac{\tan x}{x} + \log \sec^2 x\right]$

$v = (\sin x)^{\sin x}$

$\log v = \sin x \log(\sin x)$

$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{\cos x}{\sin x} + \log(\sin x)\cos x$

$= \cos x + \cos x \log(\sin x)$

$= \cos x(1 + \log \sin x)$

$\frac{dv}{dx} = v\cos x(1 + \log \sin x)$

$= \sin x^{\sin x} \cdot \cos x(1 + \log \sin x)$

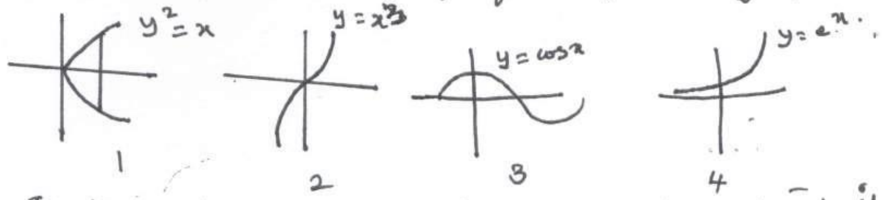
$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= x^{\tan x}\left[\frac{\tan x}{x} + \log \sec^2 x\right] + \cos x \cdot \sin x^{\sin x}(1 + \log \sin x)$

Iportant point Functions, Limits, continuity and differentiability

1. The graph of the fn is the graph of the equation $y = f(x)$

2.



If the vertical line cuts the curve at one point it is the graph of the curve

$\therefore$ 1 is not the graph of the curve.

2, 3, 4 are the graph of the curves.

3) <u>one – to – one</u>:

A function is said to be one to one if each element of the range is associated with exactly one element of the domain.

(ie) two different elements in the domain A have different images in the co-domain. The different objects in the domain have different images.

4) If the range of the function is equal to the co-domain then the function is <u>on to</u>

5) f^{-1} exists iff f is one to one and onto

6) If the function is not one-one and onto f^{-1} does not exist

7. Composition of fn ~~must~~ need not be commutative

$$f \circ g \neq g \circ f.$$

8. If f and g are two functions $f \circ g(x) = x$ and $g \circ f(x) = x$

$$f \circ g = g \circ f = I$$

9. The domain and co-domain are same of both f and g

then $f \circ g = g \circ f = I$

10 If f^{-1} exists then f is invertable

* <u>The function is a relation</u>

11. $f \circ f^{-1} = f^{-1} \circ f = I$

12. Product of two function is different from Composition of fn.

13. If the range of a function is a singleton set then the function is called constant fn

14) The graph of the even function is symmetric about y axis.

15) The graph of the odd function is symmetric about origin.

16) O + O = O

17) E + E = E

18) O + E = neither odd nor even

19) O · O = E

20) E · E = E

21) O · E = O

22) $\frac{E}{E} = E$

23) $\frac{O}{O} = E$

24) $\frac{E}{O} = O$.

Definition: Let f be function of a variable x. Let c and l be two fixed numbers. If $f(x) \to l$ as $x \to c$ then l is the limit of the function as $x \to c$.

(ie) $\lim\limits_{x \to c} f(x) = l$.

Left hand limit

If the value $x \to c$ from below or from the left

(ie) $\lim\limits_{x \to c-} f(x) = f(c)$ is known as left hand limit

Right hand limit:

If the value of $x \to c$ from greater than (or) right hand

(ie) $\lim\limits_{x \to c+} f(x) = f(c)$ is known as right hand limit.

Note If these two limits $\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c+} f(x)$ then only limit exists.

If $\lim\limits_{x \to c-} f(x) \neq \lim\limits_{x \to c+} f(x)$ then the limit does not exist.

The left and right handed limits are also known as one sided limit.

Fundamental results on limits.

1. If $f(x) = k$, $\lim_{x\to c} f(x) = k$.

2. If $f(x) = x$ $\lim_{x\to c} f(x) = c$

3) If $f(x) = kx$ $\lim_{x\to c} f(x) = k \lim_{x\to c} f(x)$

4. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

5. $\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

6. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

7. If $f(x) \le g(x)$, $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$

Some Important limits.

1) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

2) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n$

3) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

4) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

5. $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

6. $\lim_{x\to \infty} \left(1+\frac{k}{x}\right)^x = e^k$

7. $\lim_{x\to 0} (1+x)^{1/x} = e$

8) $\lim_{x\to 0} \frac{\tan x}{x} = 1$, $\lim_{x\to 0} \frac{\tan nx}{x} = n$.

9) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

Note: In the given problem the $\sqrt{\ }$ fun is given either Nr or Dr take conjugate of the fn multiply both Nr and Dr.
In the given Problem the $\sqrt{\ }$ function is given in both Nr and Dr take the conjugate of both fn and multiply both Nr and Dr by both fn.

Continuous fn

A function is said to be continuous at a point c if
1) f is well defined at $x = c$ (ie) $f(c)$ exists.
2) $\lim_{x \to c} f(x)$ exists 3) $\lim_{x \to c} f(x) = f(c)$.

1) A function is continuous in an interval $[a, b]$ if is continuous at each and every point of the interval.

2) If f, g are the two continuous fn then $f+g$, $f-g$, $f \cdot g$, are continuous at c and $g(c) \neq 0$ then $\frac{f}{g}$ is also continuous.

3) Continuous functions are functions which do not admit any break point in its graph.

4). Every polynomial function of degree n is continuous.

5. Every rational function of the form $P(x)/g(x)$ where $P(x)$ and $g(x)$ are polynomials is continuous $g(x) \neq 0$

6. The exponential function is continuous fn at all pts of R
(ie) in particular $f(x) = e^x$ is continuous.

7. The function $f(x) = \log x$ $x > 0$ is continuous at all points of R^+

8. The fn $f(x) = \sin x$ is continuous at all points of R.

9. The fn $f(x) = \cos x$ is continuous at all points of R.

10. $f(x) = |x|$ is continuous at $x=0$ but not differentiable.

11) $f(x) = \frac{|x|}{x}$ is discontinuous at $x = 0$ 12) $f(x) = x^n$ is continuous at all R.

Differentiability

Definition :1) The derivative of a given fn. $y = f(x)$ is defined as the limit of the ratio of the increment Δy of the function to the corresponding increment Δx of the independent variable.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

2) Let $y = f(x)$. As x changes from $x+h$, y changes from $f(x)$ to $f(x+h)$ then $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

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3) $\frac{d}{dx}(c) = 0$

4) $\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$

5) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

6) $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

7) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{v^2}$

8) $\frac{d}{dx}(u_1 u_2 u_3) = u_1 u_2 u_3' + u_2 u_3 u_1' + u_3 u_1 u_2'$

9) Every differentiable fn is continuous.
But the converse is not true.
(a) Every continuous fn. need not be a differentiable.

10) LHD = RHD then only it is differentiable
LHD ≠ RHD then the given fn is not differentiable.

11) $f(x) = x^{1/3}$ is not differentiable at $x=0$

12) $f(x) = |x|$ is continuous but not differentiable.

13) In particular the graph of the curve has sharp point then the function is not differentiable.

Find the domain of the rational fn. $f(x) = \frac{x^2+x+2}{x^2-x}$.

Domain of s is obtained by removing all points from R for which $g(x)=0$

$x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x = 0, 1$.

∴ The domain of s is $R - \{0, 1\}$

Examples of greatest and least integer fn.

$\lceil 2.5 \rceil = 3$ $\lceil -2.5 \rceil = -2$ $\lfloor 2.5 \rfloor = 2$ $\lfloor -2.5 \rfloor = -3$.
least integer fn Greatest integer fn