UNIT 10 OSCILLATIONS

Warm greetings:

The previous notes and videos uploaded are very useful to you. Now we are going to discuss about

- Time period,
- frequency,
- phase,
- phase difference and
- epoch in SHM
- angular simple harmonic motion

Time period, frequency, phase, phase difference and epoch in SHM:

(i) Time period:

The time period is defined as the time taken by a particle to complete one oscillation. It is usually denoted by *T*. For one complete revolution, the time taken is *t* = T, therefore

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} \quad (10.11)$$

Then, the displacement of a particle executing simple harmonic motion can be written either as sine function or cosine function.

$$y(t) = A \sin \frac{2\pi}{T} t \text{ or } y(t) = A \cos \frac{2\pi}{T} t$$

where T represents the time period. Suppose the time t is replaced by $t + T$, then the function

$$y(t+T) = A \sin \frac{2\pi}{T}(t+T)$$

$$= A \sin(\frac{2\pi}{T} t + 2\pi)$$

$$= A \sin \frac{2\pi}{T} t = y(t)$$

$$y(t+T) = y(t)$$

Thus, the function repeats after one time period. This $y(t)$ is an example of periodic function.

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(ii) Frequency and angular frequency :

The number of oscillations produced by the particle per second is called frequency. It is denoted by f. SI unit for frequency is s^{-1} or hertz (In symbol, Hz). Mathematically, frequency is related to time period by

$$f = \frac{1}{T} \quad (10.12)$$

The number of cycles (or revolutions) per second is called angular frequency. It is usually denoted by the Greek small letter 'omega', ω.

Comparing equation (10.11) and equation (10.12), angular frequency and frequency are related by

$$\omega = 2\pi f \quad (10.13)$$

SI unit for angular frequency is rad s^{-1}. (read it as radian per second)

(iii) Phase:

The phase of a vibrating particle at any instant completely specifies the state of the particle. It expresses the position and direction of motion of the particle at that instant with respect to its mean position (Figure 10.11).

$$y = A \sin(\omega t + \varphi_0) \quad (10.14)$$

where $\omega t + \varphi 0 = \varphi$ *is called the phase of the vibrating particle.*

At time $t = 0$ s (initial time), the phase $\varphi = \varphi 0$ is called epoch (initial phase) where $\varphi 0$ is called the angle of epoch.

Phase difference:

Consider two particles executing simple harmonic motions. Their equations are

$y_1 = A \sin(\omega t + \varphi_1)$ and

$y_2 = A \sin(\omega t + \varphi_2)$,

then the phase difference $\Delta\varphi = (\omega t + \varphi_2) - (\omega t + \varphi_1) = \varphi_2 - \varphi_1$.

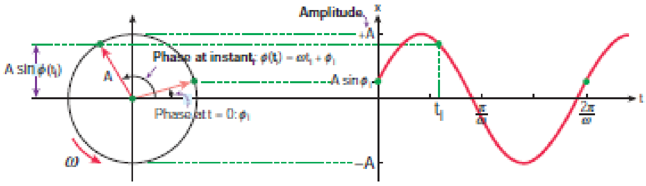

Figure 10.11 The phase of vibrating particle at two instant of time.

ANGULAR SIMPLE HARMONIC MOTION :

Time period and frequency of angular SHM :

- When a body is allowed to rotate freely about a given axis then the oscillation is known as the angular oscillation.
- The point at which the resultant torque acting on the body is taken to be zero is called mean position.
- If the body is displaced from the mean position, then the resultant torque acts such that it is proportional to the angular displacement and this torque has a tendency to bring the body towards the mean position. (Note: Torque is explained in unit 5)
- Let → be the angular displacement of the body and the resultant torque → acting on the body is

$$\vec{\tau} \propto \vec{\theta} \qquad (10.15)$$

$$\vec{\tau} = -\kappa\vec{\theta} \qquad (10.16)$$

κ is the restoring torsion constant, which is torque per unit angular displacement. If I is the moment of inertia of the body and $\vec{\alpha}$ is the angular acceleration then

$$\vec{\tau} = I\vec{\alpha} = -\kappa\vec{\theta}$$

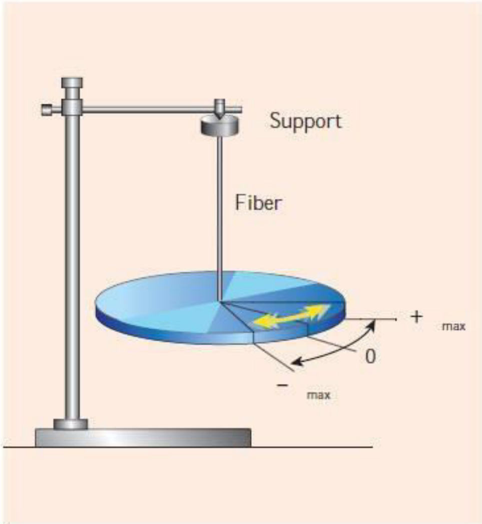

Figure 10.12 A body (disc) allowed to rotate freely about an axis

But $\vec{\alpha}=\frac{d^2\vec{\theta}}{dt^2}$ and therefore,

$$\frac{d^2\vec{\theta}}{dt^2}=-\frac{\kappa}{I}\vec{\theta} \qquad (10.17)$$

This differential equation resembles simple harmonic differential equation. So, comparing equation (10.17) with simple harmonic motion given in equation (10.10), we have

$$\omega=\sqrt{\frac{\kappa}{I}}\,rad\,s^{-1} \qquad (10.18)$$

The frequency of the angular harmonic motion is

$$f=\frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}\,Hz \qquad (10.19)$$

The time period (from equation 10.12) is

$$T=2\pi\sqrt{\frac{I}{\kappa}}\ \text{second} \qquad (10.20)$$

Comparison of Simple Harmonic Motion and Angular Simple Harmonic Motion:

In linear simple harmonic motion, the displacement of the particle is measured in terms of linear displacement $\vec{r}$. The restoring force is $\vec{F} = -k\vec{r}$, where k is a spring constant or force constant which is force per unit displacement. In this case, the inertia factor is mass of the body executing simple harmonic motion.

In angular simple harmonic motion, the displacement of the particle is measured in terms of angular displacement $\vec{\theta}$. Here, the spring factor stands for torque constant i.e., the moment of the couple to produce unit angular displacement or the restoring torque per unit angular displacement. In this case, the inertia factor stands for moment of inertia of the body executing angular simple harmonic oscillation.

Table 10.2 Comparision of simple harmonic motion and angular harmonic motion

S.No	Simple Harmonic Motion	Angular Harmonic Motion
1.	The displacement of the particle is measured in terms of linear displacement $\vec{r}$.	The displacement of the particle is measured in terms of angular displacement $\vec{\theta}$ (also known as angle of twist).
2.	Acceleration of the particle is $\vec{a} = -\omega^2 \vec{r}$	Angular acceleration of the particle is $\vec{\alpha} = -\omega^2 \vec{\theta}$.
3.	Force, $\vec{F} = m\vec{a}$, where m is called mass of the particle.	Torque, $\vec{\tau} = I\vec{\alpha}$, where I is called moment of inertia of a body.
4.	The restoring force $\vec{F} = -k\vec{r}$, where k is restoring force constant.	The restoring torque $\vec{\tau} = -\kappa\vec{\theta}$, where the symbol κ (Greek alphabet is pronounced as '*kappa*') is called restoring torsion constant. It depends on the property of a particular torsion fiber.
5.	Angular frequency, $\omega = \sqrt{\frac{k}{m}}$ rad s^{-1}	Angular frequency, $\omega = \sqrt{\frac{\kappa}{I}}$ rad s^{-1}

LINEAR SIMPLE HARMONIC OSCILLATOR (LHO) :

Horizontal oscillations of a spring-mass system:

Consider a system containing a block of mass m attached to a massless spring with stiffness constant or force constant or spring constant k placed on a smooth horizontal surface (frictionless surface) as shown in Figure 10.13.

Let x_0 be the equilibrium position or mean position of mass m when it is left undisturbed. Suppose the mass is displaced through a small displacement x towards right from its equilibrium position and then released, it will oscillate back and forth about its mean position x_0.

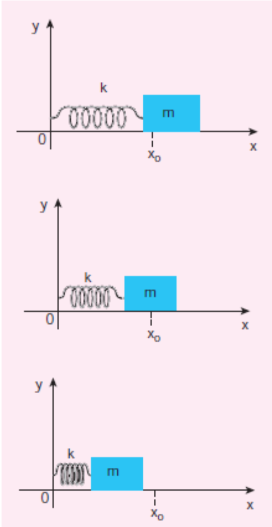

Figure 10.13 Horizontal oscillation spring-mass system

Let *F* be the restoring force (due to stretching of the spring) which is proportional to the amount of displacement of block. For one dimensional motion, mathematically, we have

$$F \propto x$$
$$F = -kx$$

Where negative sign implies that the restoring force will always act opposite to the direction of the displacement. This equation is called Hooke's law (refer to unit 7). Notice that, the restoring force is linear with the displacement (i.e., the exponent of force and displacement are unity). This is not always true; in case if we apply a very large stretching force, then the amplitude of oscillations becomes very large (which means, force is proportional to displacement containing higher powers of x) and therefore, the oscillation of the system is not linear and hence, it is called non-linear oscillation. We restrict ourselves only to linear oscillations throughout our discussions, which means Hooke's law is valid (force and displacement have a linear relationship).

From Newton's second law, we can write the equation for the particle executing simple harmonic motion

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (10.21)$$

Comparing the equation (10.21) with simple harmonic motion equation (10.10), we get

$$\omega^2 = \frac{k}{m}$$

which means the angular frequency or natural frequency of the oscillator is

$$\omega = \sqrt{\frac{k}{m}} \text{ rad } s^{-1} \qquad (10.22)$$

The frequency of the oscillation is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hertz} \qquad (10.23)$$

and the time period of the oscillation is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \text{ seconds} \qquad (10.24)$$

Notice that ***in simple harmonic motion, the time period of oscillation is independent of amplitude***. This is valid only if the amplitude of oscillation is small.

The solution of the differential equation of a SHM may be written as

$$x(t) = A\sin(\omega t + \varphi) \qquad (10.25)$$

Or

$$x(t) = A\cos(\omega t + \varphi) \qquad (10.26)$$

where A, ω and ϕ are constants. General solution for differential equation 10.21 is $x(t) = A\sin(\omega t + \varphi) + B\cos(\omega t + \varphi)$ where A and B are contants.

Note

(a) Since, mass is inertial porperty and spring constant is an elastic property,

Time period is $T=2\pi\sqrt{\frac{m}{k}}$

$$T=2\pi\sqrt{\frac{\text{Inertial property}}{\text{Elasticproperty}}}=2\pi\sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$$

(b) $\frac{\text{Displacement}}{\text{acceleration}}=\frac{x}{\frac{d^2x}{dt^2}}=-\frac{m}{k}$, whose modulus value or magnitude is $\frac{m}{k}$

hence, time period $T=2\pi\sqrt{\frac{m}{k}}$

Vertical oscillations of a spring:

Figure 10.14 Springs

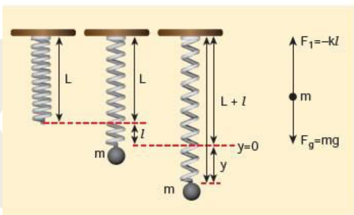

Figure 10.15 A massless spring with stiffness constant *k*

Let us consider a massless spring with stiffness constant or force constant *k* attached to a ceiling as shown in Figure 10.15. Let the length of the spring before loading mass *m* be *L*. If the block of mass *m* is attached to the other end of spring, then the spring elongates by a length *l*. Let F_1 be the restoring force due to stretching of spring. Due to mass *m*, the gravitational force acts vertically downward. We can draw free-body diagram for this system as shown in Figure 10.15. When the system is under equilibrium,

$$F_1 + mg = 0 \qquad (10.27)$$

But the spring elongates by small displacement *l*, therefore,

$$F_1 \propto l \Rightarrow F_1 = -k\,l \qquad (10.28)$$

Substituting equation (10.28) in equation (10.27), we get

$$-k\,l + mg = 0$$
$$mg = kl$$
or
$$\frac{m}{k} = \frac{l}{g} \qquad (10.29)$$

Suppose we apply a very small external force on the mass such that the mass further displaces downward by a displacement *y*, then it will oscillate up and down. Now, the restoring force due to this stretching of spring (total extension of spring is *y* + *l*) is

$$F_2 \propto (y + l)$$
$$F_2 = -k\,(y + l) = -ky - kl \qquad (10.30)$$

Since, the mass moves up and down with acceleration *d* ²*y dt*² by drawing the free body diagram for this case, we get

$$-ky - kl + mg = m\frac{d^2y}{dt^2} \qquad (10.31)$$

The net force acting on the mass due to this stretching is

$$F = F_2 + mg$$
$$F = -ky - kl + mg \qquad (10.32)$$

The gravitational force opposes the restoring force. Substituting equation (10.29) in equation (10.32), we get

$$F = -ky - kl + kl = -ky$$

Applying Newton's law, we get

$$m\frac{d^2y}{dt^2} = -k\,y$$
$$\frac{d^2y}{dt^2} = -\frac{k}{m}y \qquad (10.33)$$

The above equation is in the form of simple harmonic differential equation. Therefore, we get the time period as

$$T = 2\pi\sqrt{\frac{m}{k}}\ \text{second} \qquad (10.34)$$

The time period can be rewritten using equation (10.29)

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \text{ second} \qquad (10.35)$$

The acceleration due to gravity g can be computed from the formula

$$g = 4\pi^2\left(\frac{l}{T^2}\right) \text{ m s}^{-2} \qquad (10.36)$$

Note The time period obtained for horizontal oscillations of spring and for vertical oscillations of spring are found to be equal.

For reference:

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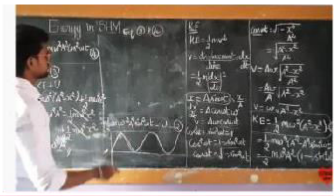

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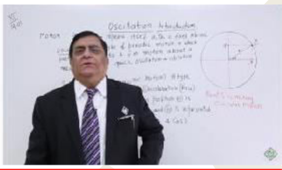

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