Chapter 8

Differentials and Partial Derivatives

Exercise Problems 8.1

Question 1.

Let f(x) = $\sqrt[3]{x}$. Find the linear approximation at x = 27. Use the linear approximation to $\sqrt[3]{27.2}$.

Solution:

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$$

we have,
$x_0 = 27$
$\Delta x = 0.2$
$x = 27.2$

$$f(x_0) = f(27) = (27)^{\frac{1}{3}} = 3$$

$$f'(x_0) = f'(27) = \frac{1}{3}(27)^{\frac{-2}{3}}$$

$$= \frac{1}{3}(3)^{-2} = \frac{1}{3}\left(\frac{1}{9}\right)$$

$$= \frac{1}{27}$$

The required linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(27.2) = 3 + \frac{1}{27}(27.2 - 27)$$

$$= 3 + \frac{1}{27}(0.2)$$

$$= 3 + 0.0074 = 3.0074 \text{ (approximately)}$$

Question 2.

Use the linear approximation to find approximate values of

(i) $(123)^{\frac{2}{3}}$ (ii) $\sqrt[4]{15}$ (iii) $\sqrt[3]{26}$

Solution:

(i) $(123)^{\frac{2}{3}} = (125 - 2)^{2/3}$

Let $f(x) = x^{2/3}$

$$f'(x) = \frac{2}{3}x^{\frac{-1}{3}}$$

we have,
$x = 123$
$x_0 = 125$
$\Delta x = -2$

$$f(x_0) = f(125) = (125)^{\frac{2}{3}} = 25$$

$$f'(x_0) = \frac{2}{3}(125)^{\frac{-1}{3}} = \frac{2}{3}\left(\frac{1}{5}\right) = \frac{2}{15}$$

The required linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$$L(x) = 25 + \frac{2}{15}(x - 125)$$

$$L(123) = 25 + \frac{2}{15}(123 - 125)$$

$$= 25 + \frac{2}{15}(-2) = 25 - \frac{4}{15}$$

$$= 25 - 0.2667$$

$$= 24.7333 \text{ (approximately)}$$

(*ii*) $\sqrt[4]{15}$

consider $(15)^{\frac{1}{4}} = (16-1)^{\frac{1}{4}}$

we have,
$x = 15$
$x_0 = 16$
$\Delta x = -1$

Let $f(x) = x^{\frac{1}{4}}$

$f'(x) = \frac{1}{4}x^{\frac{-3}{4}}$

$$f(x_0) = f(16) = (16)^{\frac{1}{4}} = 2$$

$$f'(x_0) = f'(16) = \frac{1}{4}(16)^{\frac{-3}{4}} = \frac{1}{4}\left(\frac{1}{8}\right) = \frac{1}{32}$$

The required linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$$L(x) = 2 + \frac{1}{32}(x-16)$$

$$L(15) = 2 + \frac{1}{32}(15-16)$$

$$= 2 - \frac{1}{32}$$

$$= 2 - 0.03125$$

$$= 1.96875 \text{ (approximately)}$$

(*iii*) $\sqrt[3]{26} = (26)^{\frac{1}{3}} = (27-1)^{\frac{1}{3}}$

consider $f(x) = x^{\frac{1}{3}}$

we have,
$x = 26$
$x_0 = 27$
$\Delta x = -1$

$f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$

$$f(x_0) = f(27) = (27)^{\frac{1}{3}} = 3$$

$$f'(x_0) = f'(27) = \frac{1}{3}(27)^{\frac{-2}{3}} = \frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{27}$$

The required linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$$L(x) = 3 + \frac{1}{27}(x-27)$$

$$L(26) = 3 + \frac{1}{27}(26-27) = 3 + \frac{1}{27}(-1)$$

$$= 3 - \frac{1}{27}$$

$$= 3 - 0.037$$

$$= 2.963 \text{ (approximately)}$$

Question 3.

Find a linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

(ii) $g(x) = \sqrt{x^2 + 9} + x_0 = -4$

(iii) $h(x) = \frac{x}{x+1}, x_0 = 1$

Solution:

$f(x) = x^3 - 5x + 12$

$f'(x) = 3x^2 - 5$

$f(x_0) = f(2) = (2)^3 - 5(2) + 12 = 8 - 10 + 12 = 10$

$f'(x_0) = f'(2) = 3(2)^2 - 5 = 12 - 5 = 7$

The required linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$= 10 + 7(x - 2)$

$= 10 + 7x - 14$

$= 7x - 4$

(ii)

$$g(x) = \sqrt{x^2+9}\ ;\ x_0 = -4$$

$$g'(x) = \frac{1}{2\sqrt{x^2+9}}(2x) = \frac{x}{\sqrt{x^2+9}}$$

$$g(x_0) = g(-4) = \sqrt{16+9} = \sqrt{25} = 5$$

$$g'(x_0) = \frac{-4}{\sqrt{25}} = \frac{-4}{5}$$

The required linear approximation $L(x) = g(x_0) + g'(x_0)(x - x_0)$

$$= 5 - \frac{4}{5}(x+4)$$

$$= 5 - \frac{4x}{5} - \frac{16}{5} = \frac{9}{5} - \frac{4x}{5}$$

$$= \frac{9-4x}{5}$$

(iii) $h(x) = \frac{x}{x+1}$; $x_0 = 1$

$$h'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$h'(x) = \frac{1}{(x+1)^2}$$

$$h(x_0) = h(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$h'(x_0) = h'(1) = \frac{1}{(1+1)^2} = \frac{1}{4}$$

The required linear approximation $L(x) = h(x_0) + h'(x_0)(x - x_0)$

$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4} = \frac{x}{4} + \frac{1}{4}$$

$$= \frac{x+1}{4}$$

Question 4.

The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. find the following in calculating the area of the circular plate:

(i) Absolute error

(ii) Relative error

(iii) Percentage error

Solution:

We know that Area of the circular plate $A(r) = \pi r^2$, $A'(r) = 2\pi r$

Change in Area = A′ (12.5) (0.15) = 3.75 π cm^2

Exact calculation of the change in Area = A (12.65) – A (12.5)

= 160.0225π – 156.25π

= 3.7725π cm^2

(i) Absolute error = Actual value – Approximate value

= 3.7725 π – 3.75 π

= 0.0225 π cm^2

(*ii*) Relative error $= \dfrac{\text{Actual value} - \text{Approximate value}}{\text{Actual value}}$

$= \dfrac{3.7725\,\pi - 3.75\,\pi}{3.7725\,\pi}$

$= \dfrac{0.0225\,\pi}{3.7725\,\pi}$

$= 0.006\ \text{cm}^2$

(*iii*) Percentage error $=$ Relative error $\times 100$

$= 0.006 \times 100$

$= 0.6\ \%$

Question 5.

A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9.8 cm. Find approximations for the following:

(i) change in the volume

(ii) change in the surface area

Solution:

(i) We know that Volume of sphere

$$v(r) = \frac{4}{3}\pi r^3$$

$$v'(r) = \frac{4}{3}\pi(3r^2)$$

$$v'(r) = 4\pi r^2$$

we have $r = 10$ cm

Change in volume at $r = 10$ is

$$= v'(r)\,[10 - 9.8]$$

$$= 4\pi\,(10)^2\,(0.2)$$

$$= 8\pi\ \text{cm}^3$$

∴ Volume decreases by 80π cm^3

(ii) Surface area of the sphere

$S(r) = 4\pi\, r^2$

$S'(r) = 8\pi r$

Change in surface area at r = 10 is

$= S'(r)\,[10 - 9.8]$

$= 8\pi\,(10)\,(0.2) = 16\pi\ \text{cm}^2$

∴ Surface Area decreases by 16π cm^2

Question 6.

The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation T = $2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of 1.

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$$

$$\log T = \log 2\pi + \frac{1}{2}\log\left(\frac{l}{g}\right)$$

$$\log T = \log 2\pi + \frac{1}{2}[\log l - \log g]$$

D.w.r to '*l*'

$$\frac{1}{T}\frac{dT}{dl} = 0 + \frac{1}{2}\left[\frac{1}{l} - 0\right]$$

Percentage error $\frac{dT}{T} = \frac{1}{2}\cdot\frac{1}{l}\,dl$

$$\frac{\Delta T}{T}\times 100 = \frac{1}{2}\cdot\frac{1}{l}\,dl \times 100$$

(But $dl = 0.02\,l$) $= \frac{1}{2}\left(\frac{1}{l}\right).(0.02l)\times 100$

$= 1\%$

Question 7.

Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Solution:

Let x be the number

$$\therefore y = f(x) = (x)^{\frac{1}{n}}$$

$$\log y = \frac{1}{n}\log x$$

D.w.r to '*x*'

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{n}\cdot\frac{1}{x}$$

$$\frac{dy}{y} = \frac{1}{n}\times\frac{1}{x}dx$$

Exercise 8.2

Question 1.

Find differential dy for each of the following functions:

(i) $y=\frac{(1-2x)^3}{3-4x}$ (ii) $y = (3 + \sin(2x))^{2/3}$ (iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

Solution:

(i) $y=\frac{(1-2x)^3}{3-4x}$

$$dy = \left[\frac{(3-4x)3(1-2x)^2(-2)-(1-2x)^3(-4)}{(3-4x)^2}\right]dx$$

$$(i.e.,)\quad dy = \left\{\frac{(1-2x)^2}{(3-4x)^2}\left[-6(3-4x)+4(1-2x)\right]\right\}dx$$

$$(i.e.,)\quad dy = \left\{\frac{(1-2x)^2}{(3-4x)^2}[16x-14]\right\}dx = 2\frac{(8x-7)(1-2x)^2}{(3-4x)^2}\,dx$$

$$dy = \frac{2(8x-7)(1-2x)^2}{(3-4x)^2}dx$$

(ii) $y = [3+\sin(2x)]^{\frac{2}{3}}$

$$dy = \left\{\frac{2}{3}[3+\sin(2x)]^{\frac{-1}{3}}[(\cos 2x)(2)]\right\}dx$$

$$= \left\{\frac{4}{3}(3+\sin 2x)^{\frac{-1}{3}}(\cos 2x)\right\}dx$$

$$= \left\{\frac{4}{3}\frac{\cos 2x}{(3+\sin 2x)^{\frac{1}{3}}}\right\}dx$$

(iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

$$dy = \left\{e^{x^2-5x+7}\left[-\sin(x^2-1)(2x)\right]+\cos(x^2-1)\left[e^{x^2-5x+7}(2x-5)\right]\right\}dx$$

$$dy = \left\{e^{x^2-5x+7}\left[-2x\sin(x^2-1)+(2x-5)\cos(x^2-1)\right]\right\}dx$$

$$dy = e^{x^2-5x+7}\left\{(2x-5)\cos(x^2-1)-2x\sin(x^2-1)\right\}dx$$

Question 2.

Find df for $f(x) = x^2 + 3x$ and evaluate it for

(i) x = 2 and dx = 0.1

(ii) x = 3 and dx = 0.02

Solution:

$y = f(x) = x^2 + 3x$

$dy = (2x + 3)\, dx$

(i) dy {when x = 2 and ate = 0.1} = [2(2) + 3] (0.1)

= 7(0.1) = 0.7

(ii) dy {when x = 3 and dx = 0.02} = [2(3) + 3] (0.0.2)

= 9(0.02) = 0.18

Question 3.

Find Δf and df for the function f for the indicated values of x, Δx and compare

(i) $f(x) = x^3 – 2x^2$; x = 2, Δx = dx = 0.5

(ii) $f(x) = x^2 + 2x + 3$; x = -0.5, Δx = dx = 0.1

Solution:

(i) $y = f(x) = x^3 – 2x^2$

$dy = (3x^2 – 4x)\, dx$

dy (when x = 2 and dx = 0.5) = $[3(2^2) – 4(2)] (0.5)$

= (12 – 8)(0.5) = 4(0.5) = 2

(i.e.,) df = 2

Now $\Delta f = f(x + \Delta x) – f(x)$

Here x = 2 and Δx = 0.5

$f(x) = x^3 – 2x^2$

So $f(x + \Delta x) = f(2 + 0.5) = f(2.5) = (2.5)^3 – 2 (2.5)^2 = (2.5)^2 [2.5 – 2] = 6.25 (0.5) = 3.125$

$f(x) = f(2) = 2^3 – 2(2^2) = 8 – 8 = 0$

So Δf = 3.125 – 0 = 3.125

(ii) $y = f(x) = x^2 + 2x + 3$

$dy = (2x + 2)\,dx$

dy (when $x = -0.5$ and $dx = 0.1$)

$= [2(-0.5) + 2]\,(0.1)$

$= (-1 + 2)(0.1) = (1)(0.1) = 0.1$

(i.e.,) $df = 0.1$

Now $\Delta f = f(x + \Delta x) - f(x)$

Here $x = -0.5$ and $\Delta x = 0.1$

$x^2 + 2x + 3$

$f(x + \Delta x) = f(-0.5 + 0.1) = f(-0.4)$

$= (-0.4)^2 + 2(-0.4) + 3$

$= 0.16 - 0.8 + 3 = 3.16 - 0.8 = 2.36$

$f(x) = f(-0.5) = (-0.5)^2 + 2(-0.5) + 3$

$= 0.25 - 1 + 3 = 3.25 - 1 = 2.25$

So $\Delta f = f(x + \Delta x) - f(x) = 2.36 - 2.25 = 0.11$

Question 4.

Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$.

Solution:

To find log 1003

$1003 = 1000 + 3$ and

$\log 1000 = 3$

Let $y = \log x$

$$dy = \frac{1}{x}dx$$

Here $x = 1000$ and $dx = 3 = 3 \log_{10}e$

$$\text{So } dy = \frac{1}{1000}[3 \times 0.4343] = 0.0013029$$

$$\begin{aligned} \text{Now } y + dy &= \log 1000 + dy \\ &= 3 + 0.0013029 \\ &= 3.0013029 \end{aligned}$$

Question 5.

The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

(i) Approximately, how much did the tree's diameter grow?

(ii) What is the percentage increase in area of the tree's cross-section?

Solution:

(i) Given r = 15 cm and rate of change of perimeter = 6 cm

To find the rate of change of diameter

Now perimeter = p = 2πr

So dp = 2πdr

Here dp = 6 cm (given)

$$\Rightarrow 6 = 2\pi dr$$

$$\Rightarrow \frac{6}{2\pi} = dr \text{ (i.e.,) } dr = \frac{3}{\pi}$$

(*i.e.,*) rate of change in radius $= dr = \frac{3}{\pi}$

so rate of change in diameter $= d(2r) = 2dr = 2 \times \frac{3}{\pi} = \frac{6}{\pi}$ cm

(*ii*)

$$A = \pi r^2$$

$$\text{Now } dA = \pi(2r)dr = 2\pi r dr$$

$$\text{so } \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r}$$

$$= \frac{2\left(\frac{3}{\pi}\right)}{r(=15)} = \frac{6}{15\pi} = \frac{2}{5\pi}$$

$$\text{so \% error} = \frac{dA}{A} \times 100$$

$$= \frac{2}{5\pi} \times 100 = \frac{40}{\pi}$$

Question 6.

An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm, find the volume of the shell approximately.

Solution:

$$v = \frac{4}{3}\pi r^3$$

$$dv = \frac{4}{3}\pi\left(3r^2\right)dr$$

Here $r = 5$cm and $dr = 0.3$

$$\text{so } dv = \frac{4}{3}\pi(25)\frac{3}{10}$$

$$= 30\pi \text{ mm}^3$$

Question 7.

Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

Solution:

Area of circle = A = πr^2.

$$\text{So } dA = \pi(2r)\, dr$$

$$(i.e.,) \quad dA = 2\pi r\, dr$$

(Here $r = 2$mm and $dr = 0.1$ mm)

$$\text{so } dA = 2\pi(2)(0.1) = 0.4\pi$$

Question 8.

In a newly developed city, it is estimated that the voting population (in thousands) will increase according to V (t) = $30 + 12t^2 - t^3$, $0 \le t \le 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

Solution:

$$v = 30 + 12t^2 - t^3 \qquad 0 \le t \le 8$$

$$\text{So } dv = (24t - 3t^2)\, dt$$

$$\text{Here } t = 4 \text{ and } dt = \frac{1}{6}$$

$$\text{So } dv = [24(4) - 3(4^2)]\,\frac{1}{6}$$

$$= (96 - 48)\,\frac{1}{6} = \frac{48}{6} = 8 \text{ (in thousand)} = 8000$$

Question 9.

The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}$, $0 \le x \le 9$. What is the approximate number of words learned when x changes from

(i) 1 to 1.1 hour?

(ii) 4 to 4.1 hour?

Solution:

$$y = 52\sqrt{x}$$

$$dy = 52\left(\frac{1}{2\sqrt{x}}dx\right) = \frac{26}{\sqrt{x}}dx$$

(i) dy (when $x = 1$ and $dx = 0.1$ hr)

$$= \frac{26}{\sqrt{1}}(0.1) = 2.6 \sim 3 \text{ words}$$

(ii) dy (when $x = 4$ and $dx = 0.1$ hr)

$$= \frac{26}{\sqrt{4}}(0.1) = \frac{26}{2}(0.1)$$

$$= 13(0.1) = 1.3 \sim 1 \text{ word}$$

Question 10.

A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate percentage change in the area.

Solution:

Here radius is changing from 10.5 cm to 10.75 cm

⇒ r = 10.5 cm and dr = 0.25 cm

Now area = $A = \pi r^2$

⇒ $dA = \pi (2r)\, dr$

(i) So dA

(when r = 10.5 cm and dr = 0.25 cm)

$= \pi (2 \times 10.5)(0.25)$

$= 5.25\pi$

(i) So dA

(when r = 10.5 cm and dr = 0.25 cm)

$= \pi (2 \times 10.5) (0.25)$

$= 5.25 \pi$

(*ii*) Percentage error in Area $= \frac{dA}{A} \times 100$

(*i.e.,*) Here $dA = 2\pi r\, dr$

$A = \pi r^2$

So $\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r}$

$\frac{dA}{A} \times 100$ (when $r = 10.5$ and $dr = 0.25$ cm)

$= \frac{2(0.25)}{10.5} \times 100$

$= 4.76\%$

Question 11.

A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10 cm. Use the differentials to find approximately how many cubic centimeters of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

Solution:

(i) $v = a^3$

so $dv = a^2\, da$

dv (when) a = 10 cm and da = 0.20 cm

$= 3(10^2) (0.2)$

$300 \times 0.2 = 60 \text{ cm}^3$

Actual paint used = v at $x + \Delta x = 10.2$ and $x = 10$ cm

$= a^3$ at $x + \Delta x = 10.2$ and $x = 10$

$= (10.2)^3 – (10) = 61.2 \text{ cm}^3$