479

(iv) $\frac{dy}{dx} = e^{x+y} + x^3 e^y$

$$\frac{dy}{dx} = e^x . e^y + x^3 . e^y \Rightarrow e^y(e^x + x^3)$$

$$\frac{dy}{dx} = e^y(e^x + x^3) \Rightarrow \frac{dy}{e^y} = (e^x + x^3)\,dx$$

$$\Rightarrow e^{-y} . dy = (e^x + x^3)\,dx$$

$$\Rightarrow \int e^{-y} dy = \int (e^x + x^3)\,dx$$

$$\Rightarrow -e^{-y} = e^x + \frac{x^4}{4} + c$$

$$\Rightarrow e^x + e^{-y} + \frac{x^4}{4} = -c = c$$ [$-c = c$ which is also a constant

(v) $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0.$

$$(e^y + 1)\cos x\,dx = -e^y \sin x\,dy$$

$$\frac{\cos x}{\sin x}\,dx = \frac{-e^y}{e^y + 1}\,dy$$

Put $t = e^y + 1$, $dt = e^y . dy$

$$\therefore \cot x\,dx = \frac{-dt}{t}$$

$$\Rightarrow \int \cot x\,dx = -\int \frac{dt}{t} \Rightarrow \log(\sin x) = -\log t + \log c$$

$$\Rightarrow \log \sin x + \log t = \log c$$

$$\Rightarrow \log(\sin x . t) = \log c$$

$$\Rightarrow \sin x . t = c$$ $[\because t = e^y + 1]$

$$\Rightarrow \sin x(e^y + 1) = c$$

(vi) $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = n y^2 dx.$

$$\frac{y\,dx - x\,dy}{y^2} . \cot\left(\frac{x}{y}\right) = n\,dx$$

Put $x/y = t$; $dt = \frac{y\,dx - x\,dy}{y^2}$

S.MANIKANDAN P.G.Asst in MATHEMATICS - 9655536357

EXERCISE 10.6 483

Solve the following differential equations:

1. $\left[x+y\cos\left(\frac{y}{x}\right)\right]dx = x\cos\left(\frac{y}{x}\right)dy.$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y\cos\left(\frac{y}{x}\right)}{x\cos\left(y/x\right)} \quad \ldots (1)$

This is a homogeneous differential equation

$\therefore$ put $\underline{y = vx}$

$\Rightarrow \frac{dy}{dx} = v(1) + x\frac{dv}{dx} \quad \ldots (2)$

substitute (2) in (1) we get,

$v + x\frac{dv}{dx} = \frac{x+vx\cos\left(\frac{vx}{x}\right)}{x\cos\left(\frac{vx}{x}\right)}$ $[\because y = vx]$

$= \frac{x+vx\cos v}{x\cos v} = \frac{x(1+v\cos v)}{x\cos v}$

$v + x\frac{dv}{dx} = \frac{1+v\cos v}{\cos v}$

$x\frac{dv}{dx} = \frac{1+v\cos v}{\cos v} - v = \frac{1+v\cos v - v\cos v}{\cos v}$

$x\frac{dv}{dx} = \frac{1}{\cos v}$

seperating the variables we get,

$\cos v\, dv = \frac{dx}{x}$

$\Rightarrow \int \cos v\, dv = \int \frac{dx}{x}$

$\Rightarrow \sin v = \log x + \log c$

$\Rightarrow \sin v = \log cx$

$\Rightarrow \sin\frac{y}{x} = \log cx \quad [y = vx \Rightarrow v = \frac{y}{x}]$

484

2. $(x^3+y^3)\,dy - x^2y\,dx = 0$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \ldots (1)$$

This is a homogeneous differential equation

$\therefore$ put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots (2)$

Subs. (2) in (1)

$$v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3+v^3x^3}$$

$$= \frac{x^3v}{x^3(1+v^3)} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$= \frac{v - v(1+v^3)}{1+v^3} = \frac{v-v-v^4}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

seperating variables we get,

$$\Rightarrow \quad \left(\frac{1+v^3}{v^4}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{v^4} + \frac{v^3}{v^4}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \left(v^{-4} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating.

$$\Rightarrow \quad \int\left(v^{-4} + \frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{v^{-3}}{-3} + \log v = -\log x + \log c$$

$$\Rightarrow \quad -\frac{1}{3v^3} = -\log x - \log v + \log c$$

$$\Rightarrow \quad \frac{1}{3v^3} = \log x + \log v - \log c$$

$$\Rightarrow \quad \frac{1}{3v^3} = \log\left(\frac{vx}{c}\right)$$

485

$$\Rightarrow \frac{x^3}{3y^3} = \log\left(\frac{y/x \cdot x}{c}\right) \qquad [\because v = y/x]$$

$$\Rightarrow \frac{x^3}{3y^3} = \log\left(\frac{y}{c}\right)$$

$$\Rightarrow e^{x^3/3y^3} = \frac{y}{c}$$

$$\Rightarrow y = c.e^{x^3/3y^3}$$

3. $ye^{x/y}dx = (xe^{x/y} + y)dy.$

$$\frac{dx}{dy} = \frac{xe^{x/y}+y}{ye^{x/y}} \quad \ldots (1)$$

This is a homogeneous differential equation

$\therefore$ put $\underline{x = vy}$ $\Rightarrow$ $\frac{dx}{dy} = v + y.\frac{dv}{dy}$ $\quad \ldots (2)$

subs (2) in (1),

$$\Rightarrow v + y.\frac{dv}{dy} = \frac{vy.e^v + y}{y.e^v}$$

$$= \frac{y(ve^v+1)}{y.e^v} = \frac{ve^v+1}{e^v}$$

$$\Rightarrow y.\frac{dv}{dy} = \frac{ve^v+1}{e^v} - v = \frac{ve^v+1-ve^v}{e^v}$$

$$\Rightarrow y.\frac{dv}{dy} = \frac{1}{e^v}$$

seperating the variables,

$$e^v.dv = \frac{dy}{y}$$

$$\Rightarrow \int e^v.dv = \int\frac{dy}{y}$$

$$\Rightarrow e^v = \log y + \log c$$

$$\Rightarrow e^v = \log yc$$

$$\Rightarrow e^{x/y} = \log yc. \qquad [x = vy \; ; v = x/y]$$

486

4. $2xy\,dx + (x^2+2y^2)\,dy = 0$

$\Rightarrow \quad 2xy\,dx = -(x^2+2y^2)\,dy$

$$\frac{dx}{dy} = \frac{-2xy}{x^2+2y^2}$$

This is a homogeneous differential equation.

$\therefore$ put $\underline{y = vx} \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

subs (2) in (1),

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{-2xvx}{x^2+2v^2x^2} = \frac{-2x^2v}{x^2+2v^2x^2}$$

$$= \frac{x^2(-2v)}{x^2(1+2v^2)} = \frac{-2v}{1+2v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-2v}{1+2v^2} - v$$

$$= \frac{-2v - v(1+2v^2)}{1+2v^2} = \frac{-2v-v-2v^3}{1+2v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-3v-2v^3}{1+2v^2}$$

seperating the variables,

$$\Rightarrow \quad \frac{1+2v^2}{3v+2v^3}\,dv = \frac{-dx}{x}$$

$$\Rightarrow \quad \frac{3+6v^2}{3v+2v^3}\,dv = -3\frac{dx}{x} \quad \text{[Multiply by 3]}$$

$$\Rightarrow \quad \int \frac{3+6v^2}{3v+2v^3}\,dv = -3\int \frac{dx}{x}$$

$$\Rightarrow \quad \log(3v+2v^3) = -3\log x + \log c$$

$$\Rightarrow \quad \log(3v+2v^3) = \log\left(\frac{1}{x^3}\right)\cdot c$$

$$\Rightarrow \quad 3v+2v^3 = \frac{c}{x^3}$$

$$\Rightarrow \quad 3\frac{y}{x} + 2\frac{y^3}{x^3} = \frac{c}{x^3}$$

$$\Rightarrow \quad \frac{3x^2y+2y^3}{x^3} = \frac{c}{x^3} \Rightarrow 3x^2y + 2y^3 = c.$$

5. $(y^2-2xy)\,dx = (x^2-2xy)\,dy$. 487

$\Rightarrow \frac{dy}{dx} = \frac{y^2-2xy}{x^2-2xy}$

This is a homogeneous differential equation.

$\therefore$ put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Subs. (2) in (1)

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2x^2 - 2xvx}{x^2 - 2xvx} = \frac{x^2(v^2-2v)}{x^2(1-2v)}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2-2v}{1-2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2-2v}{1-2v} - v = \frac{v^2-2v-v(1-2v)}{1-2v}$

$= \frac{v^2-2v-v+2v^2}{1-2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v^2-3v}{1-2v}$

seperating the variables we get,

$\Rightarrow \quad \frac{1-2v}{3v^2-3v}\,dv = \frac{dx}{x}$

$\Rightarrow \quad \frac{6v-3}{3v^2-3v}\,dv = -3\frac{dx}{x}$ [Multiply by -3]

$\Rightarrow \quad \int\frac{6v-3}{3v^2-3v}\,dv = -3\int\frac{dx}{x}$

$\Rightarrow \quad \log(3v^2-3v) = -3\log x + \log c$

$\Rightarrow \quad \log(3v^2-3v) = \log\left(\frac{1}{x^3}\right)\cdot c$

$\Rightarrow \quad 3v^2-3v = \frac{c}{x^3}$

$\Rightarrow \quad 3\cdot\frac{y^2}{x^2} - 3\frac{y}{x} = \frac{c}{x^3}$

$\Rightarrow \quad \frac{3xy^2 - 3x^2y}{x^3} = \frac{c}{x^3}$

$\Rightarrow \quad 3xy^2 - 3x^2y = c$

$\Rightarrow \quad xy^2 - x^2y = 3c = c.$

488

6. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y - x\cos^2(y/x)}{x} \quad \ldots (1)$$

This is a homogeneous differential equation.

$\therefore$ put $\underline{y = vx} \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots (2)$

subs (2) in (1),

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx - x\cos^2\left(\frac{vx}{x}\right)}{x} \Rightarrow \frac{vx - x\cos^2 v}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(v - \cos^2 v)}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \cos^2 v$$

$$\Rightarrow x\frac{dv}{dx} = v - \cos^2 v - v = -\cos^2 v$$

By seperating variables,

$$\Rightarrow \frac{dv}{\cos^2 v} = \frac{-dx}{x}$$

$$\Rightarrow \sec^2 v \cdot dv = -\frac{dx}{x}$$

By integrating,

$$\Rightarrow \int \sec^2 v \cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \tan v = -\log x + \log c$$

$$\Rightarrow \tan v = \log c - \log x$$

$$\Rightarrow \tan v = \log\left(\frac{c}{x}\right)$$

$$\Rightarrow e^{\tan v} = \frac{c}{x}$$

$$\Rightarrow c = x \cdot e^{\tan v} \qquad [v = y/x]$$

$$\Rightarrow c = x \cdot e^{\tan(y/x)}$$

7. $(1+3e^{y/x})dy + 3e^{y/x}(1-\frac{y}{x})dx = 0$, given that $y=0$ when $x=1$. 489

$(1+3e^{y/x})dy + 3e^{y/x}(1-\frac{y}{x})dx = 0$

$\Rightarrow (1+3e^{y/x})dy = -3e^{y/x}(1-\frac{y}{x})dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-3e^{y/x}(1-\frac{y}{x})}{1+3e^{y/x}} \quad \ldots (1)$

This is a homogeneous differential equation

$\therefore$ put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots (2)$

subs (2) in (1)

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{-3e^{vx/x}(1-\frac{vx}{x})}{1+3e^{vx/x}} \Rightarrow \frac{-3e^{v}(1-v)}{1+3e^{v}}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-3e^{v}(1-v)}{1+3e^{v}} - v$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-3e^{v} + 3ve^{v} - v(1+3e^{v})}{1+3e^{v}}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-3e^{v}+3ve^{v}-v-3ve^{v}}{1+3e^{v}} = \frac{-3e^{v}-v}{1+3e^{v}}$

$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{3e^{v}+v}{1+3e^{v}}\right)$

seperate the variables,

$\Rightarrow \quad \frac{1+3e^{v}}{3e^{v}+v}dv = -\frac{dx}{x}$

$\Rightarrow \quad \int\frac{1+3e^{v}}{3e^{v}+v}dv = -\int\frac{dx}{x}$

$\Rightarrow \quad \log(3e^{v}+v) = -\log x + \log c$

$\Rightarrow \quad \log(3e^{v}+v) = \log c - \log x$

$\Rightarrow \quad \log(3e^{v}+v) = \log\left(\frac{c}{x}\right)$

$\Rightarrow \quad 3e^{v}+v = \frac{c}{x}$

$\Rightarrow \quad 3e^{y/x} + \frac{y}{x} = \frac{c}{x}$

490

$$\Rightarrow \frac{3xe^{y/x}+y}{x} = \frac{c}{x}$$

$$\Rightarrow 3xe^{y/x}+y = c$$

when $x=1, y=0$

$$\Rightarrow 3(1)e^{0}+0 = c \quad \Rightarrow \boxed{c=3}$$

$$\therefore 3xe^{y/x}+y = 3.$$

8. $(x^2+y^2)dy = xy\,dx$. It is given that $y(1)=1$ and $y(x_0)=e$. Find the value of x_0.

$$(x^2+y^2)\,dy = xy\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad \ldots (1)$$

This is a homogenious differential equation.

$$\therefore \text{put } y=vx \Rightarrow \frac{dy}{dx} = v+x\frac{dv}{dx} \quad \ldots (2)$$

subs (2) in (1),

$$\Rightarrow v+x\frac{dv}{dx} = \frac{xvx}{x^2+v^2x^2}$$

$$= \frac{x^2v}{x^2(1+v^2)} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2}-v = \frac{v-v(1+v^2)}{1+v^2} = v-v-v^3$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

seperating the variables,

$$\Rightarrow \frac{1+v^2}{v^3}dv = \frac{-dx}{x}$$

$$\Rightarrow \frac{1}{v^3}+\frac{v^2}{v^3}\cdot dv = -\frac{dx}{x}$$

$$\Rightarrow \int v^{-3}\cdot dv + \int\frac{dv}{v} = -\int\frac{dx}{x}$$

$$\Rightarrow \frac{v^{-2}}{-2}+\log v = -\log x+\log c$$

$$\Rightarrow -\frac{1}{2v^2}+\log v = -\log x+\log c$$

492

$\Rightarrow \frac{1}{2v^2} - \log v = \log x - \log c$

$\Rightarrow \frac{1}{2v^2} = \log v + \log x - \log c$

$\Rightarrow \frac{1}{2v^2} = \log\left(\frac{vx}{c}\right)$

$\Rightarrow \frac{x^2}{2y^2} = \log\left(\frac{y}{c}\right)$ $[\because y = vx \Rightarrow v = y/x]$

$\Rightarrow e^{x^2/2y^2} = \frac{y}{c}$

$\Rightarrow y = ce^{x^2/2y^2}$

Given $y(1) = 1$

$\Rightarrow 1 = ce^{1/2(1)} \Rightarrow 1 = ce^{1/2} \Rightarrow 1 = c\sqrt{e}$

$\Rightarrow c = \frac{1}{\sqrt{e}}$

$\therefore y = \frac{1}{\sqrt{e}} \cdot e^{\frac{x^2}{2y^2}}$

Also $y(x_0) = e$

$\Rightarrow e = \frac{1}{\sqrt{e}} \cdot e^{\frac{x_0^2}{2e^2}} \Rightarrow e\sqrt{e} = e^{\frac{x_0^2}{2e^2}}$

$\Rightarrow \frac{x_0^2}{2e^2} = \log e\sqrt{e} \Rightarrow \frac{x_0^2}{2e^2} = \log e^{3/2}$

$\Rightarrow \frac{x_0^2}{2e^2} = \frac{3}{2}\log_e^e \Rightarrow \frac{x_0^2}{2e^2} = \frac{3}{2}(1)$ $[\because \log_e^e = 1]$

$\Rightarrow x_0^2 = \frac{3}{2}(2e^2) \Rightarrow x_0^2 = 3e^2$

$\Rightarrow x_0 = \pm\sqrt{3e^2} \Rightarrow x_0 = \pm\sqrt{3}.e$

$\therefore x_0 = \pm\sqrt{3}.e$

492

EXERCISE 10.7

Solve the following linear differential Equations: $\left[ye^{\int P dx} = \int Q e^{\int P dx} . dx + c \right.$

$\left. x e^{\int P dy} = \int Q e^{\int P dy} . dy + c \right]$

1. $\cos x \frac{dy}{dx} + y \sin x = 1.$

$\Rightarrow \frac{dy}{dx} + y \frac{\sin x}{\cos x} = 1/\cos x$ [divided by $\cos x$]

$\Rightarrow \frac{dy}{dx} + y \tan x = \sec x$

This is a linear differential equation

[$\because ye^{\int P dx} = \int Q e^{\int P dx} . dx + c$ when $\frac{dy}{dx} + Py = Q$]

[From the equation: $P = \tan x$; $Q = \sec x$

$\int P dx = \int \tan x . dx = \log \sec x$

$\therefore e^{\int P dx} = e^{\log \sec x} = \sec x$]

The solution is : $ye^{\int P dx} = \int Q . e^{\int P dx} . dx + c$

$\Rightarrow y \sec x = \int \sec x . \sec x . dx + c$

$\Rightarrow y \sec x = \int \sec^2 x . dx + c$

$\Rightarrow y \sec x = \tan x + c$

$\Rightarrow y . \frac{1}{\cos x} = \frac{\sin x}{\cos x} + c$

$\Rightarrow y = \sin x + c(\cos x)$

2. $(1-x^2) \frac{dy}{dx} - xy = 1.$

$\Rightarrow \frac{dy}{dx} + \left(\frac{-x}{1-x^2} \right) y = \frac{1}{1-x^2}$ [divided by $1-x^2$]

This is a linear differential equation

[$\because ye^{\int P dx} = \int Q . e^{\int P dx} . dx + c$, when $\frac{dy}{dx} + Py = Q$]

[From the equation: $P = \frac{-x}{1-x^2}$; $Q = \frac{1}{1-x^2}$

493

$\int P dx = \int \frac{-x}{1-x^2} dx = \frac{1}{2} \int \frac{-2x}{1-x^2} dx$

$= \frac{1}{2} \log(1-x^2) = \log \sqrt{1-x^2}$

$\therefore e^{\int P dx} = e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$]

$\therefore$ The solution is : $y e^{\int P dx} = \int Q . e^{\int P dx} . dx + c$

$\Rightarrow y\sqrt{1-x^2} = \int \frac{1}{1-x^2} \cdot \sqrt{1-x^2} . dx + c$

$\Rightarrow y\sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2} . \sqrt{1-x^2}} \cdot \sqrt{1-x^2} . dx + c$

$\Rightarrow y\sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}} \cdot dx + c$

$\Rightarrow y\sqrt{1-x^2} = \sin^{-1}x + c$

$\Rightarrow y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$

$\Rightarrow y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + c(1-x^2)^{-1/2}$

5. $\frac{dy}{dx} + \frac{y}{x} = \sin x$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x}(y) = \sin x$

This is a linear differential equation.

[$\therefore y e^{\int P dx} = \int Q . e^{\int P dx} dx + c$, when $\frac{dy}{dx} + Py = Q$]

From the equation : $P = \frac{1}{x}$: $Q = \sin x$

$\int P dx = \int \frac{1}{x} dx = \log x$

$\therefore e^{\int P dx} = e^{\log x} = x$

$\therefore$ The solution is : $y e^{\int P dx} = \int Q . e^{\int P dx} . dx + c$

$yx = \int \sin x \, x . dx + c$

Here using two different values

$\because u = x ; dv = \sin x$

$du = dx ; v = -\cos x$

$\therefore$ The solution is : $\int u dv = uv - \int v du$

$\Rightarrow yx = x(-\cos x) - \int -\cos x . dx$

$\Rightarrow xy = -x \cos x + \sin x . dx$

494

$\Rightarrow xy + x\cos x = \sin x + c$

$\Rightarrow x(y+\cos x) = \sin x + c$

4. $(x^2+1)\dfrac{dy}{dx} + 2xy = \sqrt{x^2+4}$.

$\Rightarrow \dfrac{dy}{dx} + \left(\dfrac{2x}{x^2+1}\right)y = \dfrac{\sqrt{x^2+4}}{x^2+1}$ [divided by x^2+1]

This is a linear differential equation.

[$\because ye^{\int P dx} = \int Q.e^{\int P dx}.dx + c$, when $\dfrac{dy}{dx} + Py = Q$]

From the equation: $P = \dfrac{2x}{x^2+1}$; $Q = \dfrac{\sqrt{x^2+4}}{x^2+1}$

$\int P dx = \int \dfrac{2x}{x^2+1}\, dx = \log(x^2+1)$

$\therefore e^{\int P dx} = e^{\log(x^2+1)} = x^2+1$

$\therefore$ The solution is : $ye^{\int P dx} = \int Q.e^{\int P dx}.dx + c$

$\Rightarrow y(x^2+1) = \int \dfrac{\sqrt{x^2+4}}{x^2+1}.(x^2+1).dx + c$

$\Rightarrow y(x^2+1) = \int \sqrt{x^2+4}.dx + c$

[$\because \int\sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c$]

$\Rightarrow y(x^2+1) = \dfrac{x}{2}\sqrt{x^2+4} + \dfrac{4}{2}\log|x+\sqrt{x^2+4}| + c$.

$\Rightarrow y(x^2+1) = \dfrac{x}{2}\sqrt{x^2+4} + 2\log|x+\sqrt{x^2+4}| + c$

5. $(2x-10y^3)dy + y\,dx = 0$.

$\Rightarrow 2x\,dy - 10y^3dy + y\,dx = 0$

$\Rightarrow 2x\,dy + y\,dx = 10y^3dy$

Multiply by y, we get

$\Rightarrow 2xy\,dy + y^2dx = 10y^4dy$ [$d(xy^2) = x.2y\,dy + y^2dx$]

$\Rightarrow d(xy^2) = 10y^4dy$

Integrating,

$\Rightarrow \int d(xy^2) = 10\int y^4dy \Rightarrow xy^2 = 10.\dfrac{y^5}{5} + c$

$\Rightarrow xy^2 = 2y^5 + c$.

495

6. $x\sin x \frac{dy}{dx} + (x\cos x + \sin x)$

$\Rightarrow \frac{dy}{dx} + \left(\frac{x\cos x + \sin x}{x\sin x}\right) = \frac{\sin x}{x\sin x} = \frac{1}{x}$

This is the linear differential equation.

$[\because ye^{\int P dx} = \int Q . e^{\int P dx} . dx + c$, when $\frac{dy}{dx} + Py = Q]$

From the equation: $P = \frac{x\cos x + \sin x}{x\sin x}$; $Q = \frac{1}{x}$

$\int P dx = \int \frac{x\cos x + \sin x}{x\sin x} . dx = \log(x\sin x)$

$\therefore e^{\int P dx} = e^{\log(x\sin x)} = x\sin x.$

$\therefore$ The solution is : $ye^{\int P dx} = \int Q . e^{\int P dx} . dx + c$

$\Rightarrow y\, x\sin x = \int \frac{1}{x} . x\sin x . dx + c$

$\Rightarrow xy\sin x = \int \sin x . dx + c$

$\Rightarrow xy\sin x = -\cos x + c$

$\Rightarrow xy\sin x + \cos x = c.$

7. $(y - e^{\sin^{-1}x})\frac{dx}{dy} + \sqrt{1-x^2} = 0.$

$\Rightarrow (y - e^{\sin^{-1}x})\frac{dx}{dy} = -\sqrt{1-x^2} \Rightarrow (y - e^{\sin^{-1}x})dx = -\sqrt{1-x^2}\,dy$

$\Rightarrow (y - e^{\sin^{-1}x})dx + \sqrt{1-x^2}\,dy = 0$

$\Rightarrow y.dx - e^{\sin^{-1}x}dx + \sqrt{1-x^2}\,dy = 0.$

Multiply by $\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$,

$\Rightarrow \frac{ye^{\sin^{-1}x}}{\sqrt{1-x^2}}dx - \frac{e^{\sin^{-1}x}.e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx + \frac{\sqrt{1-x^2}.e^{\sin^{-1}x}}{\sqrt{1-x^2}}dy = 0$

$\Rightarrow \frac{ye^{\sin^{-1}x}}{\sqrt{1-x^2}}dx - \frac{e^{2\sin^{-1}x}}{\sqrt{1-x^2}}dx + e^{\sin^{-1}x}dy = 0$

$\Rightarrow \frac{ye^{\sin^{-1}x}}{\sqrt{1-x^2}}dx + e^{\sin^{-1}x}dy = \frac{e^{2\sin^{-1}x}}{\sqrt{1-x^2}}dx$

$[\because y.\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx + e^{\sin^{-1}x}dy = d(ye^{\sin^{-1}x})]$

$\Rightarrow d(ye^{\sin^{-1}x}) = \frac{e^{2\sin^{-1}x}}{\sqrt{1-x^2}}dx.$

496

By integrating,

$$\int d(ye^{\sin^{-1}x}) = \int \frac{e^{2\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad ye^{\sin^{-1}x} = \int e^{2t}\cdot dt$$

[put $t = \sin^{-1}x \Rightarrow dt = \frac{1}{\sqrt{1-x^2}}\,dx$]

$$\Rightarrow \quad ye^{\sin^{-1}x} = \frac{e^{2t}}{2} + c$$

$$\Rightarrow \quad ye^{\sin^{-1}x} = \frac{e^{2\sin^{-1}x}}{2} + c$$

8. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1-\sqrt{x}$

This is a linear differential equation

$P = \frac{1}{(1-x)\sqrt{x}}$; $Q = 1-\sqrt{x}$

Put $\sqrt{x} = t$

$$\Rightarrow \frac{1}{2\sqrt{x}}\,dx = dt$$

$$\Rightarrow \frac{dx}{\sqrt{x}} = 2dt$$

$$\int P\,dx = \int \frac{1}{(1-x)\sqrt{x}}\,dx$$

$$= \int \frac{2dt}{1-t^2}$$

$$= 2 \times \frac{1}{2} \log\left|\frac{1+t}{1-t}\right| \quad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

$$= \log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|$$

$$\therefore e^{\int P\,dx} = e^{\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)} = \frac{1+\sqrt{x}}{1-\sqrt{x}} \quad [\because t = \sqrt{x}]$$

$\therefore$ The solution is : $ye^{\int P\,dx} = \int Q.e^{\int P\,dx}\cdot dx + c$

$$\Rightarrow y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int (1-\sqrt{x})\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) dx + c$$

$$\Rightarrow y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int (1+\sqrt{x})\,dx + c$$

$$\Rightarrow y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{x^{3/2}}{3/2} + c$$

$$\Rightarrow y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x\sqrt{x} + c.$$

9. $(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$ 497

$\Rightarrow (1+x+xy^2)\,dy+(y+y^3)\,dx=0$

$\Rightarrow (1+x+xy^2)+(y+y^3)\frac{dx}{dy}=0$

$\Rightarrow (y+y^3)\frac{dx}{dy}+(1+x+xy^2)=0$

Divided by $y+y^3$, we get

$\Rightarrow \frac{dx}{dy}+\frac{1+x+xy^2}{y+y^3}=0 \Rightarrow \frac{dx}{dy}+\frac{1+x(1+y^2)}{y+y^3}=0$

$\Rightarrow \frac{dx}{dy}+\frac{1}{y+y^3}+\frac{x(1+y^2)}{y+y^3}=0$

$\Rightarrow \frac{dx}{dy}+\frac{x(1+y^2)}{y(1+y^2)}=\frac{-1}{y(1+y^2)}$

$\Rightarrow \frac{dx}{dy}+\frac{x}{y}=\frac{-1}{y(1+y^2)}$

$\Rightarrow \frac{dx}{dy}+\frac{1}{y}(x)=\frac{-1}{y(1+y^2)}$

This a linear differential equation

$[\therefore xe^{\int P dy}=\int Q e^{\int P dy}.dy+c$, when $\frac{dx}{dy}+Px=Q]$

From the equation $P=\frac{1}{y}$, $Q=\frac{-1}{y(1+y^2)}$

$\int P dy=\int \frac{1}{y}dy=\log y$

$\therefore e^{\int P dy}=e^{\log y}=y$

$\therefore$ solution is: $xe^{\int P dy}=\int Q.e^{\int P dy}.dy+c$

$xy=\int \frac{-1}{y(1+y^2)}\cdot y\,dy+c$

$xy=-\int \frac{dy}{1+y^2}+c$

$xy=-\tan^{-1}(y)+c$

$\Rightarrow xy+\tan^{-1}(y)=c.$

498

10. $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

This is a linear differential equation

$P = \frac{1}{x\log x}$; $Q = \frac{\sin 2x}{\log x}$

$\int P dx = \int \frac{1}{x\log x} dx = \log(\log x)$

$\therefore e^{\int P dx} = e^{\log(\log x)} = \log x.$

$\therefore$ The solution is : $ye^{\int P dx} = \int Q.e^{\int P dx}.dx + c$

$\Rightarrow y\log x = \int \frac{\sin 2x}{\log x}.\log x\, dx + c$

$\Rightarrow y\log x = \int \sin 2x.dx + c$

$\Rightarrow y\log x = \frac{-\cos 2x}{2} + c$

$\Rightarrow y\log x + \frac{\cos 2x}{2} = c.$

11. $(x+a)\frac{dy}{dx} - 2y = (x+a)^4.$

$\Rightarrow (x+a)\frac{dy}{dx} - 2y = (x+a)^4$

Divided by $(x+a)$ we get,

$\Rightarrow \frac{dy}{dx} - \frac{2}{x+a}y = (x+a)^3$

This is the linear differential equation

$\therefore P = \frac{-2}{x+a}$; $Q = (x+a)^3$

$\int P dx = \int \frac{-2}{x+a} dx = -2\log(x+a). = \log\left(\frac{1}{x+a}\right)^2$

$\therefore e^{\int P dx} = e^{\log(1/x+a)^2} = \left(\frac{1}{x+a}\right)^2$

The solution is : $ye^{\int P dx} = \int Q.e^{\int P dx}.dx + c$

$\Rightarrow y\left(\frac{1}{x+a}\right)^2 = \int (x+a)^3\left(\frac{1^2}{(x+a)^2}\right) dx + c$

$\Rightarrow \left(\frac{y}{(x+a)^2}\right) = \int (x+a)^2 dx + c$

$\Rightarrow \frac{y}{(x+a)^2} = \frac{(x+a)^2}{2} + c$

$$\Rightarrow \frac{y}{(x+a)^2} = \frac{(x+a)^2+2c}{2}$$

$$\Rightarrow 2y = (x+a)^2 (x+a)^2 + 2c(x+a)^2$$

$$\Rightarrow 2y = (x+a)^4 + 2c(x+a)^2.$$

12. $\frac{dy}{dx} = \frac{\sin^2 x}{1+x^3} - \frac{3x^2y}{1+x^3}$

$\Rightarrow \frac{dy}{dx} + \frac{3x^2y}{1+x^3} = \frac{\sin^2 x}{1+x^3}$

This is a linear differential equation

$\therefore P = \frac{3x^2}{1+x^3}$; $Q = \frac{\sin^2 x}{1+x^3}$

$\int P dx = \int \frac{3x^2}{1+x^3} dx = \log(1+x^3)$

$\therefore e^{\int P dx} = e^{\log(1+x^3)} = 1+x^3$

The solution is : $y e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + c$

$$\Rightarrow y(1+x^3) = \int \frac{\sin^2 x}{1+x^3} \cdot (1+x^3) dx + c$$

$$\Rightarrow y(1+x^3) = \int \sin^2 x \cdot dx + c$$

$[\cos 2x = 1 - 2\sin^2 x \quad \therefore \sin^2 x = \frac{1-\cos 2x}{2}]$

$$\Rightarrow y(1+x^3) = \int \frac{1-\cos 2x}{2} dx + c$$

$$\Rightarrow y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

13. $x \frac{dy}{dx} + y = x \log x.$

Divide by x,

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} y = \log x$$

This is a linear differential equation

$\therefore P = \frac{1}{x}$; $Q = \log x$

$\int P dx = \int \frac{1}{x} dx = \log x$

$e^{\int P dx} = e^{\log x} = x.$

500

$\therefore$ The solution is : $ye^{\int P dx} = \int Q e^{\int P dx} . dx + c.$

$\Rightarrow yx = \int x \log x \, dx + c$

Here using two different variables

$u = \log x \; ; \; dv = x$

$du = \frac{1}{x} \; ; \; v = \frac{x^2}{2}$

$\therefore \int u dv = uv - \int v du$

$\Rightarrow yx = \log x . \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$\Rightarrow xy = \frac{x^2}{2} \log x - \frac{1}{2} \int x . dx$

$\Rightarrow xy = \frac{x^2}{2} \log x - \frac{1}{2} \cdot \frac{x^2}{2} + c$

$\Rightarrow xy = \frac{2x^2 \log x - x^2 + 4c}{4}$

$\Rightarrow 4xy = 2x^2 \log x - x^2 + 4c$

14 $x \frac{dy}{dx} + 2y - x^2 \log x = 0.$

$\Rightarrow x \frac{dy}{dx} + 2y = x^2 \log x$

divided by x,

$\Rightarrow \frac{dy}{dx} + \frac{2}{x} y = x \log x$

This is a linear differential equation.

$\therefore P = \frac{2}{x} \; ; \; Q = x \log x$

$\int P dx = \int \frac{2}{x} dx = 2 \log x = \log x^2$

$e^{\int P dx} = e^{\log x^2} = x^2.$

$\therefore$ The solution is : $ye^{\int P dx} = \int Q e^{\int P dx} . dx + c$

$\Rightarrow yx^2 = \int x \log x . (x^2) . dx + c$

$\Rightarrow x^2 y = \int x^3 \log x . dx + c$

Here using two different variables

$u = \log x \; ; \; dv = x^3$

$du = \frac{1}{x} \; ; \; v = \frac{x^4}{4}$ $\therefore \int u dv = uv - \int v du$

502

$\Rightarrow x^2y = \frac{x^4}{4}\log x - \int \frac{x^4}{4}\cdot\frac{1}{x}\cdot dx$

$\Rightarrow x^2y = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3\, dx$

$\Rightarrow x^2y = \frac{x^4}{4}\log x - \frac{1}{4}\cdot\frac{x^4}{4} + c$

$\Rightarrow x^2y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c.$

15. $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}$, given that $y = 2$ when $x = 1$.

$\Rightarrow \frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x^2}$

This is a linear differential equation.

$\therefore P = \frac{3}{x}$; $Q = \frac{1}{x^2}$

$\int P dx = \int \frac{3}{x}\, dx = 3\log x = \log x^3$

$e^{\int P dx} = e^{\log x^3} = x^3$

$\therefore$ The solution is : $ye^{\int P dx} = \int Q e^{\int P dx}\cdot dx + c$

$\Rightarrow yx^3 = \int \frac{1}{x^2}\cdot x^3\, dx + c$

$\Rightarrow yx^3 = \int x\cdot dx + c$

$\Rightarrow yx^3 = \frac{x^2}{2} + c \quad \ldots (1)$

when $x=1$, $y=2$ $\Rightarrow 2(1)^3 = \frac{1^2}{2} + c$

$\Rightarrow 2 = \frac{1}{2} + c \Rightarrow c = 2 - \frac{1}{2} \Rightarrow c = \frac{4-1}{2}$

$\Rightarrow c = 3/2$

$\therefore$ (1) becomes

$\Rightarrow yx^3 = \frac{x^2}{2} + \frac{3}{2}$

$\Rightarrow 2x^3y = x^2 + 3$

EXERCISE 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Let A be the number of bacteria at any time t.

Given $\frac{dA}{dt} \propto A$

$$\Rightarrow \frac{dA}{dt} = KA \Rightarrow \frac{dA}{A} = K\,dt$$

$$\Rightarrow \int \frac{dA}{A} = K\int dt \Rightarrow \log A = Kt + \log c \Rightarrow \log A - \log c = Kt$$

$$\Rightarrow \log\left(\frac{A}{c}\right) = Kt \Rightarrow \frac{A}{c} = e^{Kt}$$

$$\Rightarrow \underline{A = c.e^{Kt}} \quad \ldots (1)$$

Initially when $t=0$, Assume $A = A_0$

$$\therefore (1) \Rightarrow A_0 = c.e^{K(0)} \Rightarrow A_0 = c.e^0 \Rightarrow \boxed{A_0 = c}$$

In (1) subs $c = A_0$

$$\therefore \underline{A = A_0 . e^{Kt}} \quad \ldots (2)$$

(i) when $t = 5$, $A = 3A_0$

$$(2) \Rightarrow 3A_0 = A_0 e^{5K} \Rightarrow \boxed{3 = e^{5K}}$$

(ii) when $t = 10$, $A = ?$

$$(2) \Rightarrow A = A_0 e^{10K} \Rightarrow A = A_0 (e^{5K})^2 \Rightarrow A = A_0 (3)^2$$

$$\Rightarrow \boxed{A = 9A_0}$$

Hence, the amount of bacteria will be present after 10 hours is 9 times the initial amount.

2. Find the population of the city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000. 503

Let A be the number of population of city.

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = KA \Rightarrow \frac{dA}{A} = K\,dt$$

$$\Rightarrow \int \frac{dA}{A} = k\int dt \Rightarrow \log A = kt + \log c \Rightarrow \log A - \log c = kt$$

$$\Rightarrow \log\left(\frac{A}{c}\right) = kt \Rightarrow \frac{A}{c} = e^{kt} \Rightarrow A = c.e^{kt} \quad \ldots (1)$$

when $t=0$, $A = 3,00,000$

$$\therefore (1) \Rightarrow 3,00,000 = c.e^{k(0)} \Rightarrow 3,00,000 = c.e^{0} \Rightarrow c.(1)$$

$$\Rightarrow \boxed{3,00,000 = c}$$

In (1) subs $c = 3,00,000$

$$\therefore A = 3,00,000\, e^{kt} \quad \ldots (2)$$

when $t = 40$, $A = 4,00,000$

$$(2) \Rightarrow 4,00,000 = 3,00,000\, e^{40k}$$

$$\Rightarrow \frac{4}{3} = e^{40k} \Rightarrow (e^{k})^{40} = \frac{4}{3}$$

$$\Rightarrow \boxed{e^{k} = \left(\frac{4}{3}\right)^{1/40}}$$

In (2) subs $e^{k} = \left(\frac{4}{3}\right)^{1/40}$.

$$\Rightarrow \boxed{A = 3,00,000\left(\frac{4}{3}\right)^{t/40}}$$

3. The equation of electromotive force for an electric circuit containing resistance and self inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance and L the co-efficient of induction. Find the current i at time t when $E = 0$.

504

Given $E = Ri + L\frac{di}{dt}$

Written as,

$$Ri + L\frac{di}{dt} = E$$

divided by L,

$$\frac{Ri}{L} + \frac{di}{dt} = \frac{E}{L}$$

This is a linear differential equation.

$$\therefore P = \frac{R}{L}, \quad Q = \frac{E}{L}$$

$$\int P\,dt = \int \frac{R}{L}\,dt = \frac{R}{L}t$$

$$e^{\int P\,dt} = e^{\frac{Rt}{L}}.$$

$\therefore$ The solution is : $i e^{\int P dt} = \int Q \cdot e^{\int P dt}\,dt + c$

$$\Rightarrow i.e^{\frac{Rt}{L}} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}}\,dt + c$$

$$\Rightarrow i.e^{\frac{Rt}{L}} = \frac{E}{L} \cdot \frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + c$$

$$\Rightarrow i.e^{\frac{Rt}{L}} = \frac{E}{R} \cdot e^{\frac{Rt}{L}} + c$$

[divide by $e^{\frac{Rt}{L}}$] $\Rightarrow \quad i = \frac{E}{R} \cdot \frac{e^{\frac{Rt}{L}}}{e^{\frac{Rt}{L}}} + \frac{c}{e^{\frac{Rt}{L}}}$

$$\Rightarrow \quad i = \frac{E}{R} + c \cdot e^{-\frac{Rt}{L}}$$

when $E = 0$,

$$\Rightarrow i = \frac{0}{R} + c \cdot e^{-\frac{Rt}{L}}$$

$$\Rightarrow \boxed{i = ce^{-\frac{Rt}{L}}}$$

4. The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Let v be the velocity and the retardation (negative acceleration) be $-\frac{dv}{dt}$

505

Given $\frac{dv}{dt} = -v$

Seperating the variables,

$\Rightarrow \frac{dv}{v} = -dt$

$\Rightarrow \int \frac{dv}{v} = -\int dt \Rightarrow \log v = -t + \log c$

$\Rightarrow \log v - \log c = -t \Rightarrow \log\left(\frac{v}{c}\right) = -t$

$\Rightarrow \frac{v}{c} = e^{-t}$

$\Rightarrow \underline{v = c.e^{-t}} \quad \ldots (1)$

when $t=0$, $v = 10$ m/sec

$\therefore (1) \Rightarrow 10 = c.e^{0} \Rightarrow \boxed{10 = c}$

subs $c=10$ in (1)

$\therefore (1) \Rightarrow v = 10.e^{-t}$

when $t = 2$ $\Rightarrow v = 10.e^{-2} \Rightarrow \boxed{v = \frac{10}{e^2}}$

5. suppose a person deposits 10,000 Indian rupees in a bank account at a rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Let A be the principal at time t, Given ratio = 5%

$\therefore \frac{dA}{dt} = A\left(\frac{5}{100}\right) = 0.05A \Rightarrow \frac{dA}{dt} = 0.05\,A \Rightarrow \frac{dA}{A} = 0.05\,dt$

$\Rightarrow \int \frac{dA}{A} = 0.05 \int dt \Rightarrow \log A = 0.05t + \log c \Rightarrow \log A - \log c = 0.05t$

$\Rightarrow \log\left(\frac{A}{c}\right) = 0.05t \Rightarrow \frac{A}{c} = e^{0.05t} \Rightarrow \underline{A = c.e^{0.05t}} \ldots (1)$

when $t=0$, $A = 10,000$

$\therefore (1) \Rightarrow 10,000 = c.e^{0} \Rightarrow \boxed{c = 10,000}$

subs $c = 10,000$ in (1)

$\therefore (1) \Rightarrow \underline{A = 10,000\,e^{0.05t}} \quad \ldots (2)$

506

when $t = 18$ months $= 1\frac{1}{2}$ years $\therefore t = 3/2$. $A = ?$

$$(2) \Rightarrow A = 10{,}000\, e^{0.05(3/2)}$$

$$\Rightarrow \boxed{A = 10{,}000\, e^{0.075}}$$

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Let N be the radioactive nuclei in a sample at any time t

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = -KN \Rightarrow \frac{dN}{N} = -K\,dt$$

[negative sign for nuclei are decayed]

$$\Rightarrow \int \frac{dN}{N} = -K\int dt \Rightarrow \log N = -Kt + \log c \Rightarrow \log N - \log c = -Kt$$

$$\Rightarrow \log\left(\frac{N}{c}\right) = -Kt \Rightarrow \frac{N}{c} = e^{-Kt} \Rightarrow \underline{N = ce^{-Kt}} \quad \ldots (1)$$

Let Assume $t = 0$, $N = N_0$

$$\therefore (1) \Rightarrow N_0 = c.e^{-K(0)} \Rightarrow \boxed{N_0 = c}$$

Subs $c = N_0$ in (1)

$$\therefore (1) = \underline{N = N_0 e^{-Kt}} \quad \ldots (2)$$

(i) when $t = 100$, $N = 90\%\, N_0 = \frac{90}{100} N_0$

[$\because$ 10% gets disintegrated]

$$\therefore (2) \Rightarrow \frac{90}{100} N_0 = N_0 e^{-K(100)} \Rightarrow \frac{9}{10} = (e^{-K})^{100}$$

$$\Rightarrow \boxed{e^{-K} = \left(\frac{9}{10}\right)^{1/100}}$$

Subs e^{-K} in (2)

$$\therefore (2) \Rightarrow N = N_0 (e^{-K})^{t} \Rightarrow N = N_0 \left(\frac{9}{10}\right)^{\frac{1}{100}\cdot t} \quad \ldots (3)$$

507

(ii) when $t=1000$, $N=?$

$$\Rightarrow \therefore (3) \Rightarrow N = N_0 \left(\frac{9}{10}\right)^{\frac{1}{100}\times 1000}$$

$$\Rightarrow N = N_0\left(\frac{9}{10}\right)^{10} \Rightarrow N = N_0\left(\frac{9^{10}}{10^{10}}\right)$$

To find the percentage

$$\Rightarrow N = N_0\left(\frac{9^{10}}{10^8 \times \cancel{10^2}} \times \cancel{100}\right)$$

$$\Rightarrow \boxed{N = N_0\left(\frac{9^{10}}{10^8}\%\right)}$$

Hence, $\frac{9^{10}}{10^8}\%$ of radioactive nuclei will remain after 1000 years.

7. Water at temperature $100^\circ c$ cools in 10 minutes to $80^\circ c$ in a room temperature of $25^\circ c$.

Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is $40^\circ c$

$[\log_e \frac{11}{15} = -0.3101 ;\ \log_e 5 = 1.6094]$

Let T be the temperature of water at any time t.

S be the room temperature

Given $S = 25^\circ c$

$$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = K(T-S) \Rightarrow \frac{dT}{T-S} = K\,dt \Rightarrow \int\frac{dT}{T-S} = K\int dt$$

$$\Rightarrow \log(T-S) = Kt + \log c \Rightarrow \log(T-S) - \log c = Kt$$

$$\Rightarrow \log\left(\frac{T-S}{c}\right) = Kt \Rightarrow \frac{T-S}{c} = e^{Kt} \Rightarrow T-S = c\cdot e^{Kt}$$

$$\Rightarrow \underline{T-25^\circ = c\cdot e^{Kt}} \ldots (1)$$

when $t=0$, $T=100$

$$\therefore (1) \Rightarrow 100^\circ - 25^\circ = c\cdot e^{0K} \Rightarrow \boxed{75^\circ = c(1)}$$

subs $c=75$ in (1)

$$\underline{T-25^\circ = 75^\circ\cdot e^{Kt}} \ldots (2)$$

508

when $t=10$, $T=80^\circ$

$\therefore (2) \Rightarrow 80^\circ - 25^\circ = 75^\circ e^{10K}$

$\Rightarrow 55^\circ = 75^\circ e^{10K} \Rightarrow e^{10K} = \frac{55}{75}$

$\Rightarrow \boxed{e^{10K} = \frac{11}{15}}$

(i) when $t = 20$ min, $T = ?$

$\therefore (2) \Rightarrow T - 25^\circ = 75^\circ e^{20K}$

$\Rightarrow T - 25^\circ = 75^\circ (e^{10K})^2$

$\Rightarrow T - 25^\circ = 75^\circ \left(\frac{11}{15}\right)^2$

$\Rightarrow T - 25^\circ = \cancel{75}^{\,3} \times \frac{11}{\cancel{15}} \times \frac{11}{\cancel{15}_{\,3}}$

$\Rightarrow T - 25^\circ = \frac{121}{3} \Rightarrow T = \frac{121}{3} + 25$

$\Rightarrow T = 40.33 + 25$

$\Rightarrow \boxed{T = 65.33^\circ c}$

Also, $e^{10K} = \frac{11}{15}$

$\log e^{10K} = \log\left(\frac{11}{15}\right) \Rightarrow 10K = -0.3101$

$\Rightarrow \boxed{K = -0.03101}$

(ii) when $T = 40^\circ c$, $t = ?$

$\therefore (2) \Rightarrow 40 - 25 = 75 e^{kt}$

$\Rightarrow 15 = 75 e^{kt} \Rightarrow \frac{15}{75} = e^{kt} \Rightarrow \frac{1}{5} = e^{kt}$

$\Rightarrow e^{-kt} = 5 \Rightarrow -kt = \log 5$

$\Rightarrow \log 5 = 0.03101\, t$

$\Rightarrow t = \frac{\log 5}{0.03101} \Rightarrow t = \frac{1.6094}{0.03101}$

$\Rightarrow \boxed{t = 51.89 \text{ minutes.}}$

8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby kitchen counter to cool. At this instant the temperature of the coffee was 180°F and 10 minutes later it was 160°F. Assume that constant temperature of the kitchen was 70°F. 509

(i) What was the temperature of the coffee at 10.15 A.M?
(ii) The woman likes to drink coffee when its temperature is between 130°F and 140°F between what times should be have drunk the coffee?

Let T be the temperature of the coffee at time t.
S be the temperature of the kitchen. Given S=70°F

$$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = K(T-S) \Rightarrow \frac{dT}{T-S} = K\,dt \Rightarrow \int\frac{dT}{T-S} = K\int dt$$

$$\Rightarrow \log(T-S) = Kt + \log c \Rightarrow \log(T-S) - \log c = Kt$$

$$\Rightarrow \log\left(\frac{T-S}{c}\right) = Kt \Rightarrow \frac{T-S}{c} = e^{Kt} \Rightarrow T-S = c.e^{Kt}$$

$$\Rightarrow T-70° = c.e^{Kt} \ldots (1)$$

when $t=0$, $T=180°F$

$$\Rightarrow 180°-70° = ce^{0(K)} \Rightarrow \boxed{110° = c(1)}$$

Subs $c=110°$ in (1)

$$\Rightarrow T-70° = 110°e^{Kt} \ldots (2)$$

when $t=10$, $T=160°F$

$$\therefore (2) \Rightarrow 160°-70° = 110°e^{K(10)} \Rightarrow 90 = 110e^{10K}$$

$$\Rightarrow e^{10K} = \frac{90}{110} \Rightarrow \boxed{\frac{90}{110} = e^{10K}} \qquad \left[\because \sqrt{\frac{9}{11}} = 0.9045\right]$$

(i) when $t=15$, $T=?$

$$\therefore (2) = T-70 = 110e^{15K} \Rightarrow T-70 = 110e^{(30/2)K} \Rightarrow T-70 = 110\left(e^{(10K)3/2}\right)$$

$$\Rightarrow T-70 = 110\left(\frac{90}{110}\right)\left(\frac{90}{110}\right)^{1/2} \Rightarrow T-70 = 90\left(9/11\right)^{1/2}$$

$$\Rightarrow T-70 = 90(0.9045) \Rightarrow T-70 = 81.4.$$

510

$\Rightarrow T = 81.4 + 70$

$\Rightarrow T = 151.4^{\circ}F$

Also,

$$e^{10k} = \frac{9}{11} \Rightarrow \log e^{10k} = \log \frac{9}{11} \Rightarrow 10k = -0.2006$$

$$\Rightarrow \cancel{10}k = \frac{-0.2006}{10} \Rightarrow \boxed{k = -0.02006.}$$

(ii) $T = 130^{\circ}, t = ?$

$$\therefore (2) \Rightarrow 130 - 70 = 110e^{kt} \Rightarrow 60 = 110e^{kt}$$

$$\Rightarrow \frac{60}{110} = e^{kt} \Rightarrow \log \frac{6}{11} = kt$$

$$\Rightarrow -0.606135 = -0.02006\, t$$

$$\Rightarrow \frac{+0.606135}{0.02006} = t \Rightarrow \boxed{t = 30.216 \text{ min}}$$

$T = 140^{\circ}, t = ?$

$$\therefore (2) \Rightarrow 140 - 70 = 110e^{kt} \Rightarrow 70 = 110e^{kt}$$

$$\Rightarrow \frac{70}{110} = e^{kt} \Rightarrow \log \frac{7}{11} = kt$$

$$\Rightarrow -0.45198 = -0.02006t$$

$$\Rightarrow \frac{0.452}{0.02006} = t \Rightarrow \boxed{t = 22.53 \text{ min}}$$

$\therefore$ The woman should drunk the tea between ~~22~~ 10.22 to 10.30 minutes.

9. A pot of boiling water at $100^{\circ}c$ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to $80^{\circ}c$, and another 5 minutes later it has dropped to $65^{\circ}c$. Determine the temperature of the kitchen.

Let T be the temperature of boiling water at time t

S be the temperature of kitchen

$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = k(T-S) \Rightarrow \frac{dT}{T-S} = k\,dt \Rightarrow \int \frac{dT}{T-S} = k\int dt$

$\Rightarrow \log(T-S) = kt + \log c \Rightarrow \log(T-S) - \log c = kt$

$\Rightarrow \log\left(\frac{T-S}{c}\right) = kt \Rightarrow \frac{T-S}{c} = e^{kt} \Rightarrow T-S = c.e^{kt} \ldots (1)$

when $t=0, T=100^\circ$

$\therefore (1) \Rightarrow 100-S = c.e^{k(0)} \Rightarrow \boxed{100-S = c}$

Subs. $c = 100-S$ in (1)

$\Rightarrow T-S = (100-S)e^{kt} \ldots (2)$

when $t=5, T=80^\circ$

$\therefore (2) \Rightarrow 80-S = (100-S)e^{5k} \Rightarrow \boxed{e^{5k} = \frac{80-S}{100-S}}$

when $t=10, T=65^\circ$

$\therefore (2) = 65-S = (100-S)e^{10k} \Rightarrow 65-S = (100-S)(e^{5k})^2$

$\Rightarrow 65-S = (\cancel{100-S})\left(\frac{80-S}{\cancel{100-S}} \times \frac{80-S}{100-S}\right)$

$\Rightarrow 65-S(100-S) = 80-S \times 80-S$

$\Rightarrow 6500 - 165S + S^2 = 6400 - 160S + S^2$

$\Rightarrow 6500 - 6400 = 165S - 160S$

$\Rightarrow 100 = 5S \Rightarrow S = \frac{100}{5} \Rightarrow \boxed{S = 20^\circ}$

$\therefore$ kitchen temperature is 20° C

10. A tank initially contains 50 litres of pure water. Starting at time t=0 a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same

rate. Find the amount of salt present in the tank at any time $t>0$. 512

Let $x(t)$ denote the amount of salt in the tank at time t.

Its rate of change is

$$\frac{dx}{dt} \text{ (inflow rate - outflow rate)}$$

Now, 2 gram in 3 litres per minute,

inflow rate = 6 grams of salt $(3\times 2=6)$

Outflow $= \frac{3}{50}$ times of $x = \frac{3}{50}x$

$$\therefore \frac{dx}{dt} = 6 - \frac{3}{50}x \Rightarrow \frac{300-3x}{50}$$

$$\therefore \frac{dx}{dt} = -\frac{3}{50}(x-100)$$

$$\Rightarrow \frac{dx}{x-100} = \frac{-3}{50}dt \Rightarrow \int\frac{dx}{x-100} = \frac{-3}{50}\int dt$$

$$\Rightarrow \log(x-100) = -\frac{3}{50}t + \log c \Rightarrow \log(x-100) - \log c = \frac{-3}{50}t$$

$$\Rightarrow \log\left(\frac{x-100}{c}\right) = \frac{-3t}{50} \Rightarrow \frac{x-100}{c} = e^{-3t/50}$$

$$\Rightarrow x-100 = c.e^{-3t/50} \quad \ldots (1)$$

when $t=0,\ x=0$

$$\Rightarrow 0-100 = ce^{-3(0)/50} \Rightarrow -100 = ce^{0}$$

$$\Rightarrow \boxed{-100 = c}$$

Subs. $c=-100$ in (1)

$$\Rightarrow x-100 = -100.e^{-3t/50}$$

$$\Rightarrow x = 100-100e^{-3t/50}$$

$$\Rightarrow \boxed{x = 100(1-e^{-3t/50})}.$$