MATRICES AND DETERMINANTS

EXAMPLES

Example 1.1

If $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)A = |A| I_3$.

Solution

We find that $|A| = \begin{vmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{vmatrix} = 8(21-16)+6(-18+8)+2(24-14)=40-60+20=0.$

$$\text{adj } A=\begin{bmatrix}(21-16) & -(-18+8) & (24-14)\\ -(-18+8) & (24-4) & -(-32+12)\\ (24-14) & -(-32+12) & (56-36)\end{bmatrix}^T=\begin{bmatrix}5 & 10 & 10\\ 10 & 20 & 20\\ 10 & 20 & 20\end{bmatrix}.$$

So, we get

$$A(\text{adj } A) = \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\begin{bmatrix}5 & 10 & 10\\ 10 & 20 & 20\\ 10 & 20 & 20\end{bmatrix}$$

$$= \begin{bmatrix}40-60+20 & 80-120+40 & 80-120+40\\ -30+70-40 & -60+140-80 & -60+140-80\\ 10-40+30 & 20-80+60 & 20-80+60\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0I_3=|A|I_3,$$

Similarly, we get

$$(\text{adj } A)A = \begin{bmatrix}5 & 10 & 10\\ 10 & 20 & 20\\ 10 & 20 & 20\end{bmatrix}\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$$

$$= \begin{bmatrix}40-60+20 & -30+70-40 & 10-40+30\\ 80-120+40 & -60+140-80 & 20-80+60\\ 80-120+40 & -60+140-80 & 20-80+60\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0I_3=|A|I_3.$$

Hence, $A(\text{adj } A) = (\text{adj } A)A = |A|I_3.$ ■

Example 1.2

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is non-singular, find A^{-1}.

Solution

We first find adj A. By definition, we get $\text{adj } A=\begin{bmatrix}+M_{11} & -M_{12}\\ -M_{21} & +M_{22}\end{bmatrix}^T=\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}^T=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$

Since A is non-singular, $|A| = ad-bc \neq 0$.

As $A^{-1}=\frac{1}{|A|}\text{adj } A$, we get $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$ ■

Example 1.3

Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$.

Solution

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}. \text{ Then } |A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 2(7) + (-12) + 3(-1) = -1 \neq 0.$$

Therefore, A^{-1} exists. Now, we get

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} & +\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -3 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -5 & 3 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = \frac{1}{(-1)}\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}.$$

Example 1.4

If A is a non-singular matrix of odd order, prove that $|\operatorname{adj} A|$ is positive.

Solution

Let A be a non-singular matrix of order $2m+1$, where $m = 0, 1, 2, \cdots$. Then, we get $|A| \neq 0$ and, by theorem 1.9 (ii), we have $|\operatorname{adj} A| = |A|^{(2m+1)-1} = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that $|\operatorname{adj} A|$ is positive. ■

Example 1.5

Find a matrix A if $\operatorname{adj}(A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$.

Solution

$$\text{First, we find } |\operatorname{adj}(A)| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix} = 7(77-35) - 7(-7-77) - 7(-5-121) = 1764 > 0.$$

So, we get

$$A = \pm\frac{1}{\sqrt{|\operatorname{adj} A|}}\operatorname{adj}(\operatorname{adj} A) = \pm\frac{1}{\sqrt{1764}}\begin{bmatrix} +(77-35) & -(-7-77) & +(-5-121) \\ -(49+35) & +(49+77) & -(35-77) \\ +(49+77) & -(49-7) & +(77+7) \end{bmatrix}^T$$

$$= \pm\frac{1}{42}\begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}^T = \pm\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Example 1.6

If $\text{adj } A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

Solution

We compute $|\text{adj } A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 9$.

So, we get $A^{-1} = \pm\frac{1}{\sqrt{|\text{adj}(A)|}}\text{adj}(A) = \pm\frac{1}{\sqrt{9}}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \pm\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Example1.7

If A is symmetric, prove that adj A is also symmetric.

Solution

Suppose A is symmetric. Then, $A^T = A$ and so, by theorem 1.9 (vi), we get
$\text{adj}(A^T) = (\text{adj } A)^T \Rightarrow \text{adj } A = (\text{adj } A)^T \Rightarrow$ adj A is symmetric.

Example 1.8

Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$.

Solution

For the given A, we get $|A| = (2)(7) - (9)(1) = 14 - 9 = 5$. So, $A^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} & -\frac{9}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$.

Then, $(A^{-1})^T = \begin{bmatrix} \frac{7}{5} & -\frac{1}{5} \\ -\frac{9}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$. ... (1)

For the given A, We get $A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$. So $|A^T| = (2)(7) - (1)(9) = 5$.

Then, $(A^T)^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$. ... (2)

From (1) and (2), we get $(A^{-1})^T = (A^T)^{-1}$. Thus, we have verified the given property.

Example 1.9

Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$.

Solution

We get $AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$

$$(AB)^{-1} = \frac{1}{(0+6)}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad \ldots (1)$$

$$A^{-1} = \frac{1}{(0+3)}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{(2-0)}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}. \quad \ldots (2)$$

As the matrices in (1) and (2) are same, $(AB)^{-1} = B^{-1}A^{-1}$ is verified. ■

Example 1.10

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$. Hence, find A^{-1}.

Solution

$$\text{Since } A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix},$$

$$A^2 + xA + yI_2 = O_2 \Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22+4x+y & 27+3x \\ 18+2x & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, we get $22+4x+y=0, 31+5x+y=0, 27+3x=0$ and $18+2x=0$.

Hence $x=-9$ and $y=14$. Then, we get $A^2 - 9A + 14I_2 = O_2$.

Post-multiplying this equation by A^{-1}, we get $A - 9I_2 + 14A^{-1} = O_2$. Hence, we get

$$A^{-1} = \frac{1}{14}(9I_2 - A) = \frac{1}{14}\left(9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}.$$

Example 1.11

Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution

Let $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. Then, $A^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^T=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$.

So, we get

$$AA^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ \sin\theta\cos\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$$

Similarly, we get $A^TA=I_2$. Hence $AA^T=A^TA=I_2 \Rightarrow A$ is orthogonal.

Example 1.12

If $A=\frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Solution

If A is orthogonal, then $AA^T=A^TA=I_3$. So, we have

$$AA^T=I_3 \Rightarrow \frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}\frac{1}{7}\begin{bmatrix}6 & b & 2\\ -3 & -2 & c\\ a & 6 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}45+a^2 & 6b+6+6a & 12-3c+3a\\ 6b+6+6a & b^2+40 & 2b-2c+18\\ 12-3c+3a & 2b-2c+18 & c^2+13\end{bmatrix}=49\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{Bmatrix}45+a^2=49\\ b^2+40=49\\ c^2+13=49\\ 6b+6+6a=0\\ 12-3c+3a=0\\ 2b-2c+18=0\end{Bmatrix} \Rightarrow \begin{Bmatrix}a^2=4, b^2=9, c^2=36,\\ a+b=-1, a-c=-4, b-c=-9\end{Bmatrix} \Rightarrow a=2, b=-3, c=6$$

So, we get $A=\frac{1}{7}\begin{bmatrix}6 & -3 & 2\\ -3 & -2 & 6\\ 2 & 6 & 3\end{bmatrix}$ and hence, $A^{-1}=A^T=\frac{1}{7}\begin{bmatrix}6 & -3 & 2\\ -3 & -2 & 6\\ 2 & 6 & 3\end{bmatrix}$.

EXERCISE 1.1

1. Find the adjoint of the following:

(i) $\begin{bmatrix}-3 & 4\\ 6 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$

2. Find the inverse (if it exists) of the following:

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

3. If $F(\alpha)=\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1}=F(-\alpha)$.

4. If $A=\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2-3A-7I_2=O_2$. Hence find A^{-1}

5. If $A=\frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1}=A^T$.

6. If $A=\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)A=|A|I_2$.

7. If $A=\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

8. If $\text{adj}(A)=\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

9. If $\text{adj}(A)=\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}.

10. Find $\text{adj}(\text{adj}(A))$ if $\text{adj } A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

11. $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1}=\begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

13. Given $A=\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB=C$.

14. If $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1}=\frac{1}{2}(A^2-3I)$.

Example 1.13

Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

Solution

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2/8} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ This is also a row-echelon form of the given matrix.

So, a row-echelon form of a matrix is not necessarily unique.

Example 1.14

Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row-echelon form.

Solution

$$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 4R_1} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{2}{3}R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & \frac{22}{3} & 16 \end{bmatrix} \xrightarrow{R_3 \to 3R_3} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix}$$

Example 1.15

Find the rank of each of the following matrices: (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Solution

(i) Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$. Then A is a matrix of order 3×3. So $\rho(A) \le \min\{3,3\} = 3$. The highest order of minors of A is 3. There is only one third order minor of A.

It is $\begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 3(6-6) - 2(6-6) + 5(3-3) = 0$. So, $\rho(A) < 3$.

Next consider the second-order minors of A.

We find that the second order minor $\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \ne 0$. So $\rho(A) = 2$.

(ii) Let $A=\begin{bmatrix}4 & 3 & 1 & -2\\ -3 & -1 & -2 & 4\\ 6 & 7 & -1 & 2\end{bmatrix}$. Then A is a matrix of order 3×4. So $\rho(A)\le \min\{3,4\}=3$.

The highest order of minors of A is 3. We search for a non-zero third-order minor of A. But we find that all of them vanish. In fact, we have

$$\begin{vmatrix}4 & 3 & 1\\ -3 & -1 & -2\\ 6 & 7 & -1\end{vmatrix}=0;\ \begin{vmatrix}4 & 3 & -2\\ -3 & -1 & 4\\ 6 & 7 & 2\end{vmatrix}=0;\ \begin{vmatrix}4 & 1 & -2\\ -3 & -2 & 4\\ 6 & -1 & 2\end{vmatrix}=0;\ \begin{vmatrix}3 & 1 & -2\\ -1 & -2 & 4\\ 7 & -1 & 2\end{vmatrix}=0.$$

So, $\rho(A)<3$. Next, we search for a non-zero second-order minor of A.

We find that $\begin{vmatrix}4 & 3\\ -3 & -1\end{vmatrix}=-4+9=5\ne 0$. So, $\rho(A)=2$.

Example 1.16

Find the rank of the following matrices which are in row-echelon form :

(i) $\begin{bmatrix}2 & 0 & -7\\ 0 & 3 & 1\\ 0 & 0 & 1\end{bmatrix}$ (ii) $\begin{bmatrix}-2 & 2 & -1\\ 0 & 5 & 1\\ 0 & 0 & 0\end{bmatrix}$ (iii) $\begin{bmatrix}6 & 0 & -9\\ 0 & 2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

Solution

(i) Let $A=\begin{bmatrix}2 & 0 & -7\\ 0 & 3 & 1\\ 0 & 0 & 1\end{bmatrix}$. Then A is a matrix of order 3×3 and $\rho(A)\le 3$

The third order minor $|A|=\begin{vmatrix}2 & 0 & -7\\ 0 & 3 & 1\\ 0 & 0 & 1\end{vmatrix}=(2)(3)(1)=6\ne 0$. So, $\rho(A)=3$.

Note that there are three non-zero rows.

(ii) Let $A=\begin{bmatrix}0 & 5 & 1\\ 0 & 0 & 0\end{bmatrix}$. Then A is a matrix of order 3×3 and $\rho(A)\le 3$.

The only third order minor is $|A|=\begin{vmatrix}-2 & 2 & -1\\ 0 & 5 & 1\\ 0 & 0 & 0\end{vmatrix}=(-2)(5)(0)=0$. So $\rho(A)\le 2$.

There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix}-2 & 2\\ 0 & 5\end{vmatrix}=(-2)(5)=-10\ne 0$. So, $\rho(A)=2$.

Note that there are two non-zero rows. The third row is a zero row.

(iii) Let $A=\begin{bmatrix}6 & 0 & -9\\ 0 & 2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. Then A is a matrix of order 4×3 and $\rho(A)\le 3$.

The last two rows are zero rows. There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix}6 & 0\\ 0 & 2\end{vmatrix}=(6)(2)=12\ne 0$. So, $\rho(A)=2$.

Note that there are two non-zero rows. The third and fourth rows are zero rows.

(iii) Let $A=\begin{bmatrix}6&0&-9\\0&2&0\\0&0&0\\0&0&0\end{bmatrix}$. Then A is a matrix of order 4×3 and $\rho(A)\le3$.

The last two rows are zero rows. There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix}6&0\\0&2\end{vmatrix}=(6)(2)=12\ne0$. So, $\rho(A)=2$.

Note that there are two non-zero rows. The third and fourth rows are zero rows.

We observe from the above example that the rank of a matrix in row echelon form is equal to the number of non-zero rows in it. We state this observation as a theorem without proof. ■

Example 1.17

Find the rank of the matrix $\begin{bmatrix}1&2&3\\2&1&4\\3&0&5\end{bmatrix}$ by reducing it to a row-echelon form.

Solution

Let $A=\begin{bmatrix}1&2&3\\2&1&4\\3&0&5\end{bmatrix}$. Applying elementary row operations, we get

$$A\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1}\begin{bmatrix}1&2&3\\0&-3&-2\\0&-6&-4\end{bmatrix}\xrightarrow{R_3\to R_3-2R_2}\begin{bmatrix}1&2&3\\0&-3&-2\\0&0&0\end{bmatrix}.$$

The last equivalent matrix is in row-echelon form. It has two non-zero rows. So, $\rho(A)=2$.

Example 1.18

Find the rank of the matrix $\begin{bmatrix}2&-2&4&3\\-3&4&-2&-1\\6&2&-1&7\end{bmatrix}$ by reducing it to an echelon form.

Solution

Let A be the matrix. Performing elementary row operations, we get

$$A=\begin{bmatrix}2&-2&4&3\\-3&4&-2&-1\\6&2&-1&7\end{bmatrix}\xrightarrow{R_2\to2R_2}\begin{bmatrix}2&-2&4&3\\-6&8&-4&-2\\6&2&-1&7\end{bmatrix}\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2+3R_1}\begin{bmatrix}2&-2&4&3\\0&2&8&7\\0&8&-13&-2\end{bmatrix}.$$

$$\xrightarrow{R_3\to R_3-4R_2}\begin{bmatrix}2&-2&4&3\\0&2&8&7\\0&0&-45&-30\end{bmatrix}\xrightarrow{R_3\to R_3\div(-15)}\begin{bmatrix}2&-2&4&3\\0&2&8&7\\0&0&3&2\end{bmatrix}.$$ ■

The last equivalent matrix is in row-echelon form. It has three non-zero rows. So, $\rho(A)=3$.

Elementary row operations on a matrix can be performed by pre-multiplying the given matrix by a special class of matrices called elementary matrices.

Example 1.19

Show that the matrix $\begin{bmatrix}3 & 1 & 4\\ 2 & 0 & -1\\ 5 & 2 & 1\end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Solution

Let $A=\begin{bmatrix}3 & 1 & 4\\ 2 & 0 & -1\\ 5 & 2 & 1\end{bmatrix}$. Then, $|A| = 3(0+2)-1(2+5)+4(4-0) = 6-7+16 = 15 \neq 0$. So, A is non-singular. Keeping the identity matrix as our goal, we perform the row operations sequentially on A as follows:

$$\begin{bmatrix}3 & 1 & 4\\ 2 & 0 & -1\\ 5 & 2 & 1\end{bmatrix} \xrightarrow{R_1\to\frac{1}{3}R_1} \begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\ 2 & 0 & -1\\ 5 & 2 & 1\end{bmatrix} \xrightarrow{R_2\to R_2-2R_1,\ R_3\to R_3-5R_1} \begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\ 0 & -\frac{2}{3} & -\frac{11}{3}\\ 0 & \frac{1}{3} & -\frac{17}{3}\end{bmatrix} \xrightarrow{R_2\to\left(-\frac{3}{2}\right)R_2} \begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\ 0 & 1 & \frac{11}{2}\\ 0 & \frac{1}{3} & -\frac{17}{3}\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\frac{1}{3}R_2,\ R_3\to R_3-\frac{1}{3}R_2} \begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{11}{2}\\ 0 & 0 & -\frac{15}{2}\end{bmatrix} \xrightarrow{R_3\to\left(-\frac{2}{15}\right)R_3} \begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{11}{2}\\ 0 & 0 & 1\end{bmatrix} \xrightarrow{R_1\to R_1+\frac{1}{2}R_3,\ R_2\to R_2-\frac{11}{2}R_3} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}-1 & 3\\ 4 & -7\\ 3 & -4\end{bmatrix}$ (iii) $\begin{bmatrix}1 & -2 & -1 & 0\\ 3 & -6 & -3 & 1\end{bmatrix}$ (iv) $\begin{bmatrix}1 & -2 & 3\\ 2 & 4 & -6\\ 5 & 1 & -1\end{bmatrix}$ (v) $\begin{bmatrix}0 & 1 & 2 & 1\\ 0 & 2 & 4 & 3\\ 8 & 1 & 0 & 2\end{bmatrix}$

2. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 1 & -2 & 3\\ 1 & -1 & 1\end{bmatrix}$ (iii) $\begin{bmatrix}3 & -8 & 5 & 2\\ 2 & -5 & 1 & 4\\ -1 & 2 & 3 & -2\end{bmatrix}$

Example 1.22

Solve the following system of linear equations, using matrix inversion method:
$5x+2y=3,\ 3x+2y=5$.

Solution

The matrix form of the system is $AX = B$, where $A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}3\\ 5\end{bmatrix}$.

We find $|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10-6 = 4 \neq 0$. So, A^{-1} exists and $A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$.

Then, applying the formula $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} \frac{-4}{4} \\ \frac{16}{4} \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

So the solution is $(x = -1, y = 4)$.

Example 1.23

Solve the following system of equations, using matrix inversion method:

$2x_1 + 3x_2 + 3x_3 = 5, \quad x_1 - 2x_2 + x_3 = -4, \quad 3x_1 - x_2 - 2x_3 = 3.$

Solution

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

We find $|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1) - 3(-2-3) + 3(-1+6) = 10+15+15 = 40 \neq 0$.

So, A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{40}\begin{bmatrix} +(4+1) & -(-2-3) & +(-1+6) \\ -(-6+3) & +(-4-9) & -(-2-9) \\ +(3+6) & -(2-3) & +(-4-3) \end{bmatrix}^T = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Then, applying $X = A^{-1}B$, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

So, the solution is $(x_1 = 1, x_2 = 2, x_3 = -1)$. ■

Example 1.24

If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$.

Solution

We find $AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$

and $BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$.

So, we get $AB = BA = 8I_3$. That is, $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$. Hence, $B^{-1} = \frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}. \text{ That is, } B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}.$$

$$\text{So, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Hence, the solution is $(x = 3, y = -2, z = -1)$.

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:
   (i) $2x + 5y = -2, \ \ x + 2y = -3$ (ii) $2x - y = 8, \ \ 3x + 2y = -2$
   (iii) $2x + 3y - z = 9, \ x + y + z = 9, \ 3x - y - z = -1$
   (iv) $x + y + z - 2 = 0, \ 6x - 4y + 5z - 31 = 0, \ 5x + 2y + 2z = 13$
2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$.
3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)
4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.
5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Example 1.25

Solve, by Cramer's rule, the system of equations

$x_1 - x_2 = 3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7.$

Solution

First we evaluate the determinants

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 6 \neq 0, \Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 12, \Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = -6, \Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 24.$$

By Cramer's rule, we get $x_1 = \frac{\Delta_1}{\Delta} = \frac{12}{6} = 2,\ x_2 = \frac{\Delta_2}{\Delta} = \frac{-6}{6} = -1,\ x_3 = \frac{24}{6} = 4.$

So, the solution is $(x_1 = 2, x_2 = -1, x_3 = 4)$.

Example 1.26

In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points $(10,8),(20,16),(30,18)$, can you conclude that Chennai Super Kings won the match?

Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is $(70,0)$.)

Solution

The path $y = ax^2 + bx + c$ passes through the points $(10,8),(20,16),(40,22)$. So, we get the system of equations $100a + 10b + c = 8, 400a + 20b + c = 16, 1600a + 40b + c = 22$. To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 1000 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1000[-2+12-16] = -6000,$$

$$\Delta_1 = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 20 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix} = 20[-8+3+10] = 100,$$

$$\Delta_2 = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 200 \begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix} = 200[-3+48-84] = -7800,$$

$$\Delta_3 = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 2000 \begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix} = 2000[-10+84-64] = 20000.$$

By Cramer's rule, we get $a = \frac{\Delta_1}{\Delta} = -\frac{1}{60}, b = \frac{\Delta_2}{\Delta} = \frac{7800}{6000} = \frac{78}{60} = \frac{13}{10}, c = \frac{\Delta_3}{\Delta} = -\frac{20000}{6000} = -\frac{20}{6} = -\frac{10}{3}$.

So, the equation of the path is $y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$.

When $x = 70$, we get $y = 6$. So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

   (i) $5x - 2y + 16 = 0, x + 3y - 7 = 0$

   (ii) $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

   (iii) $3x + 3y - z = 11,\ 2x - y + 2z = 9,\ 4x + 3y + 2z = 25$

   (iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of the two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Example 1.27

Solve the following system of linear equations, by Gaussian elimination method :

$4x + 3y + 6z = 25,\ x + 5y + 7z = 13,\ 2x + 9y + z = 1.$

Solution

Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1,} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div (-1)]{R_2 \to R_2 \div (-1),} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 1 & 13 & 25 \end{array}\right] \xrightarrow{R_3 \to 17R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{array}\right].$$

The equivalent system is written by using the echelon form:

$$x+5y+7z = 13, \quad \ldots (1)$$
$$17y+22z = 27, \quad \ldots (2)$$
$$199z = 398. \quad \ldots (3)$$

From (3), we get $z=\frac{398}{199}=2$.

Substituting $z=2$ in (2), we get $y=\frac{27-22\times 2}{17}=\frac{-17}{17}=-1$.

Substituting $z=2, y=-1$ in (1), we get $x=13-5\times(-1)-7\times 2=4$.

So, the solution is $(x=4, y=-1, z=2)$.

Note. The above method of going from the last equation to the first equation is called the **method of back substitution.**

Example 1.28

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t)=at^2+bt+c,\ 0\le t\le 100$ where $a,b,$ and c are constants. It has been found that the speed at times $t=3, t=6$, and $t=9$ seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time $t=15$ seconds. (Use Gaussian elimination method.)

Solution

Since $v(3)=64,\ v(6)=133,$ and $v(9)=208$, we get the following system of linear equations

$$9a+3b+c = 64,$$
$$36a+6b+c = 133,$$
$$81a+9b+c = 208.$$

We solve the above system of linear equations by Gaussian elimination method.

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operations, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c}9&3&1&64\\36&6&1&133\\81&9&1&208\end{array}\right]\xrightarrow{R_2\to R_2-4R_1,R_3\to R_3-9R_1}\left[\begin{array}{ccc|c}9&3&1&64\\0&-6&-3&-123\\0&-18&-8&-368\end{array}\right]\xrightarrow{R_2\to R_2\div(-3),R_3\to R_3\div(-2)}\left[\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&9&4&184\end{array}\right]$$

$$\xrightarrow{R_3\to 2R_3}\left[\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&18&8&368\end{array}\right]\xrightarrow{R_3\to R_3-9R_2}\left[\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&0&-1&-1\end{array}\right]\xrightarrow{R_3\to(-1)R_3}\left[\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&0&1&1\end{array}\right].$$

Writing the equivalent equations from the row-echelon matrix, we get

$9a+3b+c=64,\ 2b+c=41,\ c=1.$

Writing the equivalent equations from the row-echelon matrix, we get

$9a+3b+c=64,\ 2b+c=41,\ c=1.$

By back substitution, we get $c=1,\ b=\frac{(41-c)}{2}=\frac{(41-1)}{2}=20,\ a=\frac{64-3b-c}{9}=\frac{64-60-1}{9}=\frac{1}{3}$.

So, we get $v(t)=\frac{1}{3}t^2+20t+1$. Hence, $v(15)=\frac{1}{3}(225)+20(15)+1=75+300+1=376.$

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

   (i) $2x-2y+3z=2,\quad x+2y-z=3,\quad 3x-y+2z=1$

   (ii) $2x+4y+6z=22,\quad 3x+8y+5z=27,\quad -x+y+2z=2$

2. If ax^2+bx+c is divided by $x+3, x-5$, and $x-1$, the remainders are $21, 61$ and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)

3. An amount of ₹ 65,000 is invested in three bonds at the rates of $6\%, 8\%$ and 10% per annum respectively. The total annual income is ₹ 4,800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

4. A boy is walking along the path $y=ax^2+bx+c$ through the points $(-6,8), (-2,-12)$, and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method.)