VINAYAKAR MATRICULATION HIGHER SECONDARY SCHOOL, SIVAKASI.

12th Standard - MATHS

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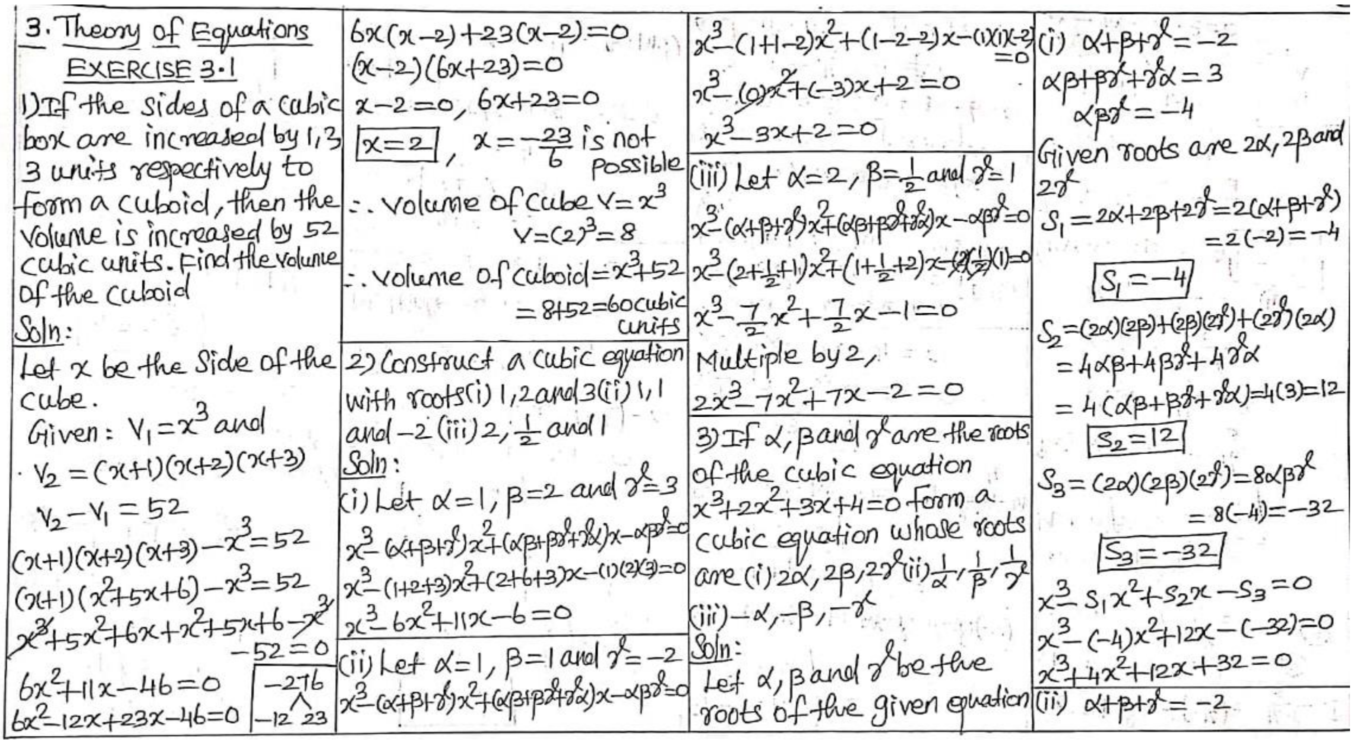
THEORY of Equation – Exercise Problems



3. Theory of Equations

EXERCISE 3.1

1) If the sides of a cubic box are increased by 1, 3 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid

Soln:

Let x be the side of the cube.

Given: $V_1 = x^3$ and

$V_2 = (x+1)(x+2)(x+3)$

$V_2 - V_1 = 52$

$(x+1)(x+2)(x+3) - x^3 = 52$

$(x+1)(x^2+5x+6) - x^3 = 52$

$\cancel{x^3}+5x^2+6x+x^2+5x+6-\cancel{x^3}-52=0$

$6x^2+11x-46=0$

$6x^2-12x+23x-46=0$

$$\begin{array}{c} -276 \\ \wedge \\ -12 \quad 23 \end{array}$$

$6x(x-2)+23(x-2)=0$

$(x-2)(6x+23)=0$

$x-2=0,\ 6x+23=0$

$\boxed{x=2}$, $x = -\frac{23}{6}$ is not possible

$\therefore$ Volume of cube $V = x^3$

$V = (2)^3 = 8$

$\therefore$ Volume of cuboid $= x^3+52$

$= 8+52 = 60$ cubic units

2) Construct a cubic equation with roots (i) 1, 2 and 3 (ii) 1, 1 and −2 (iii) 2, $\frac{1}{2}$ and 1

Soln:

(i) Let $\alpha = 1, \beta = 2$ and $\gamma = 3$

$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$

$x^3-(1+2+3)x^2+(2+6+3)x-(1)(2)(3)=0$

$x^3-6x^2+11x-6=0$

(ii) Let $\alpha=1, \beta=1$ and $\gamma=-2$

$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$

$x^3-(1+1-2)x^2+(1-2-2)x-(1)(1)(-2)=0$

$x^3-\cancel{(0)x^2}+(-3)x+2=0$

$x^3-3x+2=0$

(iii) Let $\alpha=2, \beta=\frac{1}{2}$ and $\gamma=1$

$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$

$x^3-(2+\frac{1}{2}+1)x^2+(1+\frac{1}{2}+2)x-\cancel{(2)}(\frac{1}{\cancel{2}})(1)=0$

$x^3-\frac{7}{2}x^2+\frac{7}{2}x-1=0$

Multiple by 2,

$2x^3-7x^2+7x-2=0$

3) If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$ form a cubic equation whose roots are (i) $2\alpha, 2\beta, 2\gamma$ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (iii) $-\alpha, -\beta, -\gamma$

Soln:

Let α, β and γ be the roots of the given equation

(i) $\alpha+\beta+\gamma=-2$

$\alpha\beta+\beta\gamma+\gamma\alpha=3$

$\alpha\beta\gamma=-4$

Given roots are $2\alpha, 2\beta$ and 2γ

$S_1 = 2\alpha+2\beta+2\gamma = 2(\alpha+\beta+\gamma)$

$= 2(-2) = -4$

$\boxed{S_1=-4}$

$S_2=(2\alpha)(2\beta)+(2\beta)(2\gamma)+(2\gamma)(2\alpha)$

$=4\alpha\beta+4\beta\gamma+4\gamma\alpha$

$=4(\alpha\beta+\beta\gamma+\gamma\alpha)=4(3)=12$

$\boxed{S_2=12}$

$S_3=(2\alpha)(2\beta)(2\gamma)=8\alpha\beta\gamma$

$=8(-4)=-32$

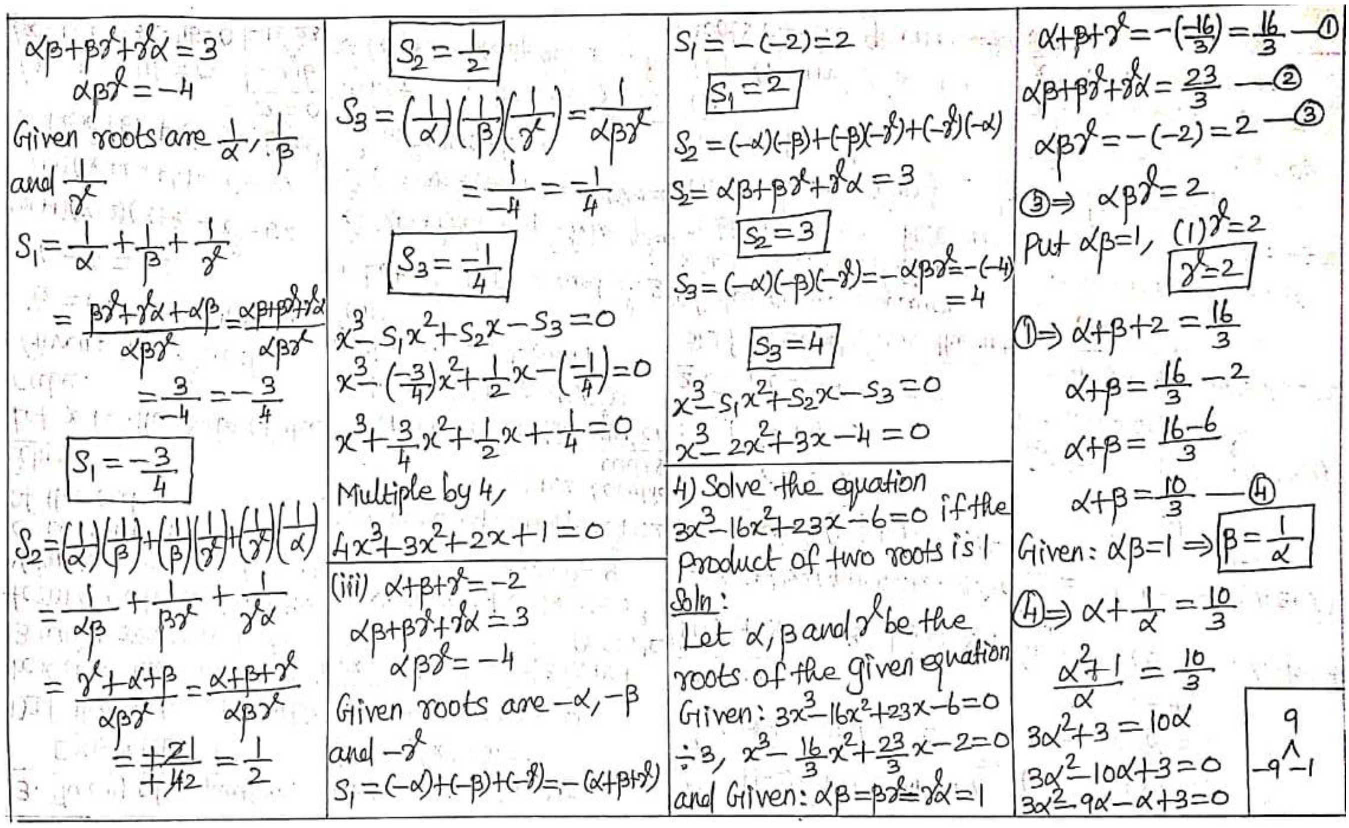
$\boxed{S_3=-32}$

$x^3-S_1x^2+S_2x-S_3=0$

$x^3-(-4)x^2+12x-(-32)=0$

$x^3+4x^2+12x+32=0$

(ii) $\alpha+\beta+\gamma=-2$



$\alpha\beta+\beta\gamma+\gamma\alpha=3$

$\alpha\beta\gamma=-4$

Given roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$

$S_1=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$

$=\frac{3}{-4}=-\frac{3}{4}$

$\boxed{S_1=-\frac{3}{4}}$

$S_2=\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)+\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)+\left(\frac{1}{\gamma}\right)\left(\frac{1}{\alpha}\right)$

$=\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}$

$=\frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$=\frac{\cancel{+2}1}{\cancel{+4}2}=\frac{1}{2}$

$\boxed{S_2=\frac{1}{2}}$

$S_3=\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)=\frac{1}{\alpha\beta\gamma}$

$=\frac{1}{-4}=\frac{-1}{4}$

$\boxed{S_3=\frac{-1}{4}}$

$x^3-S_1x^2+S_2x-S_3=0$

$x^3-\left(\frac{-3}{4}\right)x^2+\frac{1}{2}x-\left(\frac{-1}{4}\right)=0$

$x^3+\frac{3}{4}x^2+\frac{1}{2}x+\frac{1}{4}=0$

Multiple by 4,

$4x^3+3x^2+2x+1=0$

(iii) $\alpha+\beta+\gamma=-2$

$\alpha\beta+\beta\gamma+\gamma\alpha=3$

$\alpha\beta\gamma=-4$

Given roots are $-\alpha, -\beta$ and $-\gamma$

$S_1=(-\alpha)+(-\beta)+(-\gamma)=-(\alpha+\beta+\gamma)$

$S_1=-(-2)=2$

$\boxed{S_1=2}$

$S_2=(-\alpha)(-\beta)+(-\beta)(-\gamma)+(-\gamma)(-\alpha)$

$S_2=\alpha\beta+\beta\gamma+\gamma\alpha=3$

$\boxed{S_2=3}$

$S_3=(-\alpha)(-\beta)(-\gamma)=-\alpha\beta\gamma=-(-4)$

$=4$

$\boxed{S_3=4}$

$x^3-S_1x^2+S_2x-S_3=0$

$x^3-2x^2+3x-4=0$

4) Solve the equation $3x^3-16x^2+23x-6=0$ if the product of two roots is 1

Soln:

Let α, β and γ be the roots of the given equation

Given: $3x^3-16x^2+23x-6=0$

$\div 3,\ x^3-\frac{16}{3}x^2+\frac{23}{3}x-2=0$

and Given: $\alpha\beta=\beta\gamma=\gamma\alpha=1$

$\alpha+\beta+\gamma=-\left(\frac{-16}{3}\right)=\frac{16}{3}$ — ①

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{23}{3}$ — ②

$\alpha\beta\gamma=-(-2)=2$ — ③

③ $\Rightarrow \alpha\beta\gamma=2$

Put $\alpha\beta=1$, $(1)\gamma=2$

$\boxed{\gamma=2}$

① $\Rightarrow \alpha+\beta+2=\frac{16}{3}$

$\alpha+\beta=\frac{16}{3}-2$

$\alpha+\beta=\frac{16-6}{3}$

$\alpha+\beta=\frac{10}{3}$ — ④

Given: $\alpha\beta=1 \Rightarrow \boxed{\beta=\frac{1}{\alpha}}$

④ $\Rightarrow \alpha+\frac{1}{\alpha}=\frac{10}{3}$

$\frac{\alpha^2+1}{\alpha}=\frac{10}{3}$

$3\alpha^2+3=10\alpha$

$3\alpha^2-10\alpha+3=0$

$3\alpha^2-9\alpha-\alpha+3=0$

$$\begin{array}{c} 9 \\ \wedge \\ -9 \quad -1 \end{array}$$

Let p and q are the roots of the given equation.

Given: $lx^2+nx+n=0$

$\div l$, $x^2+\frac{n}{l}x+\frac{n}{l}=0$

$p+q=\frac{-n}{l}$ and

$pq=\frac{n}{l}$

$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}$

$=\frac{\sqrt{p}}{\sqrt{q}}+\frac{\sqrt{q}}{\sqrt{p}}+\frac{\sqrt{n}}{\sqrt{l}}$

$=\frac{p+q}{\sqrt{pq}}+\frac{\sqrt{n}}{\sqrt{l}}$

$=\frac{-\cancel{\frac{n}{l}}\sqrt{\frac{n}{l}}}{\sqrt{\cancel{\frac{n}{l}}}}+\frac{\sqrt{n}}{\sqrt{l}}$

$=-\frac{\sqrt{n}}{\sqrt{l}}+\frac{\sqrt{n}}{\sqrt{l}}=0$

$\therefore \sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$

10) If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$

Soln:

Let α be the common root of the given equations

Put $x=\alpha$,

$\alpha^2+p\alpha+q=0$ — ①

$\alpha^2+p'\alpha+q'=0$ — ②

① ⇒ $\cancel{\alpha^2}+p\alpha+q=0$

② ⇒ $\underset{-}{\cancel{\alpha^2}}\underset{-}{+}p'\alpha\underset{-}{+}q'=0$

$p\alpha-p'\alpha+q-q'=0$

$p\alpha-p'\alpha=-q+q'$

$\alpha(p-p')=-q+q'$

$\alpha=\frac{-q+q'}{p-p'}$

$\alpha=\frac{+(q-q')}{+(p'-p)}$

$\boxed{\alpha=\frac{q-q'}{p'-p}}$

① × p' ⇒ $\alpha^2p'+\cancel{pp'\alpha}+p'q=0$

② × p ⇒ $\underset{-}{\alpha^2p}\underset{-}{+}\cancel{pp'\alpha}\underset{-}{+}pq'=0$

$\alpha^2p'-\alpha^2p+p'q-pq'=0$

$\alpha^2p'-\alpha^2p=pq'-p'q$

$\alpha^2(p'-p)=pq'-p'q$

$\alpha^2=\frac{pq'-p'q}{p'-p}$

$\alpha\cdot\alpha=\frac{pq'-p'q}{p'-p}$

$\alpha=\frac{pq'-p'q}{\alpha(p'-p)}$

$=\frac{pq'-p'q}{\left(\frac{q-q'}{\cancel{p'-p}}\right)\cancel{(p'-p)}}$

$\alpha=\frac{pq'-p'q}{q-q'}$

$\therefore \alpha=\frac{pq'-p'q}{q-q'}$

(or) $\alpha=\frac{q-q'}{p'-p}$

10) [OR]

Let α be the common root of the given equations

Put $x=\alpha$,

$\alpha^2+p\alpha+q=0$ and

$\alpha^2+p'\alpha+q'=0$

By cross multiplication

$$\begin{array}{cccc} & \alpha^2 & \alpha & 1 \\ p & q & 1 & p \\ p' & q' & 1 & p' \end{array}$$

$\frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'}=\frac{1}{p'-p}$

$\frac{\alpha^{\cancel{2}}}{pq'-p'q}=\frac{\cancel{\alpha}}{q-q'}$

$\boxed{\alpha=\frac{pq'-p'q}{q-q'}}$

and $\frac{\alpha}{q-q'}=\frac{1}{p'-p}$

$\boxed{\alpha=\frac{q-q'}{p'-p}}$

$\therefore \alpha=\frac{pq'-p'q}{q-q'}$

(or) $\alpha=\frac{q-q'}{p'-p}$

11) Formulate into a mathematical problem to find a number such that when its cube root is added to it, the result is 6

Soln:

Let x be the number

Given: $x^{1/3}+x=6$

$x^{1/3}=6-x$

$(x^{1/\cancel{3}})^{\cancel{3}}=(6-x)^3$

$x=(6)^3-3(6)^2x+3(6)x^2-x^3$

$x=216-108x+18x^2-x^3$

$x^3-18x^2+108x+x-216=0$

$x^3-18x^2+109x-216=0$

12) A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was cut away.

Soln:

Let x be the length of the part which is standing and y be the part which was cut away

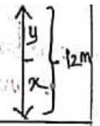
$\therefore x+y=12 \Rightarrow y=12-x$

Also, $x=(y)^{1/3} \Rightarrow x^3=(y^{1/\cancel{3}})^{\cancel{3}}$

$x^3=y$

$x^3=12-x$

$x^3+x-12=0$



Ex 3.1

If α and β are the roots of the quadratic equation $17x^2+43x-73=0$ construct a quadratic equation whose roots are $\alpha+2$ and $\beta+2$

Soln:

Let α and β are the roots of the given equation

Given: $17x^2+43x-73=0$

$\div 17$, $x^2+\frac{43}{17}x-\frac{73}{17}=0$

$S_1=\alpha+\beta=-\frac{43}{17}$

$S_2=\alpha\beta=-\frac{73}{17}$

Given roots are $\alpha+2$ and $\beta+2$

equation $ax^3+bx^2+cx+d=0$ find the Value of $\Sigma\frac{\alpha}{\beta\gamma}$ in terms of the Coefficients.

Soln:

Let α, β and γ be the roots of the given equa-tion

Given: $ax^3+bx^2+cx+d=0$

$\div a$, $x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$

$S_1=\alpha+\beta+\gamma=-\frac{b}{a}$

$S_2=\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$

$S_3=\alpha\beta\gamma=-\frac{d}{a}$

$\Sigma\frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}$

$=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$

$=\frac{\frac{c}{\cancel{a}}}{-\frac{d}{\cancel{a}}}=-\frac{c}{d}$

$\therefore\Sigma\frac{1}{\alpha}=-\frac{c}{d}$

and $\Sigma\frac{\alpha}{\beta\gamma}=\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}$

$=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$

$=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$

$=\frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{-\frac{d}{a}}$

$=\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)\left(\frac{a}{-d}\right)$

$=\left(\frac{b^2-2ac}{a^{\cancel{2}}}\right)\left(\frac{-\cancel{a}}{d}\right)$

$=\frac{2ac-b^2}{ad}$

$\therefore\Sigma\frac{\alpha}{\beta\gamma}=\frac{2ac-b^2}{ad}$

8) If α,β,γ and δ are the roots of the polynomial equation $2x^4+5x^3-7x^2+8=0$ Find a quadratic equation with integer Coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

Soln:

Let α,β,γ and δ be the roots of the given equation

Given: $2x^4+5x^3-7x^2+8=0$

$\div 2$, $x^4+\frac{5}{2}x^3-\frac{7}{2}x^2+4=0$

$\Rightarrow x^4+\frac{5}{2}x^3-\frac{7}{2}x^2+0x+4=0$

$\alpha+\beta+\gamma+\delta=-\frac{5}{2}$ and

$\alpha\beta\gamma\delta=4$

$S.R=(\alpha+\beta+\gamma+\delta)+(\alpha\beta\gamma\delta)$

$=-\frac{5}{2}+4$

$S.R=\frac{3}{2}$

$P.R=(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$

$=\left(-\frac{5}{\cancel{2}}\right)(\cancel{4}^{2})=-10$

$P.R=-10$

$\therefore$ The required equation is $x^2-(S.R)x+P.R=0$

$x^2-\frac{3}{2}x-10=0$

Multiple by 2,

$2x^2-3x-20=0$

9) If p and q are the roots of the equation $lx^2+nx+n=0$ show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$

Soln:

$\alpha\left(\frac{1}{\lambda}+1+\lambda\right)=\frac{-b}{a}$ —①

$\left(\frac{\alpha}{\lambda}\right)(\alpha)+(\alpha)(\alpha\lambda)+(\alpha\lambda)\left(\frac{\alpha}{\lambda}\right)=\frac{c}{a}$

$\frac{\alpha^2}{\lambda}+\alpha^2\lambda+\alpha^2=\frac{c}{a}$

$\alpha^2\left(\frac{1}{\lambda}+\lambda+1\right)=\frac{c}{a}$

$\alpha^2\left(\frac{1}{\lambda}+1+\lambda\right)=\frac{c}{a}$ —②

$\left(\frac{\alpha}{\cancel{\lambda}}\right)(\alpha)(\alpha\cancel{\lambda})=-\frac{d}{a}$

$\alpha^3=-\frac{d}{a}$ —③

$\frac{②}{①}\Rightarrow\frac{\alpha^{\cancel{2}}\cancel{\left(\frac{1}{\lambda}+1+\lambda\right)}}{\cancel{\alpha}\cancel{\left(\frac{1}{\lambda}+1+\lambda\right)}}=\frac{\frac{c}{\cancel{a}}}{\frac{-b}{\cancel{a}}}$

$\boxed{\alpha=-\frac{c}{b}}$

$③\Rightarrow\left(\frac{-c}{b}\right)^3=\frac{-d}{a}$

$\frac{\cancel{-}c^3}{b^3}=\frac{\cancel{-}d}{a}$

$ac^3=db^3$

EX 3.21

If the roots of $x^3+px^2+qx+r=0$ are in H.P. Prove that $9pqr=27r^2+2p$

Soln:

Let the roots be in H.P Then their reciprocals are in A.P

$\left(\frac{1}{x}\right)^3+p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)+r=0$

$\frac{1}{x^3}+p\left(\frac{1}{x^2}\right)+q\left(\frac{1}{x}\right)+r=0$

Multiple by x^3, we get,

$1+px+qx^2+rx^3=0$

$\Rightarrow rx^3+qx^2+px+1=0$ —①

$\therefore$ The roots are $\alpha-d$, α, $\alpha+d$

$(\alpha-\cancel{d})+\alpha+(\alpha+\cancel{d})=-\frac{q}{r}$

$3\alpha=-\frac{q}{r}$

$\boxed{\alpha=-\frac{q}{3r}}$

$\therefore\alpha$ is a root of ①, we get, Put $x=\alpha$,

$r\alpha^3+q\alpha^2+p\alpha+1=0$

$r\left(\frac{-q}{3r}\right)^3+q\left(\frac{-q}{3r}\right)^2+p\left(\frac{-q}{3r}\right)+1=0$

$\frac{-\cancel{r}q^3}{27r^{\cancel{3}2}}+\frac{q^3}{9r^2}-\frac{pq}{3r}+1=0$

$\frac{-q^3}{27r^2}+\frac{q^3}{9r^2}-\frac{pq}{3r}+1=0$

Multiple by $27r^2$,

$-q^3+3q^3-9pqr+27r^2=0$

$2q^3-9pqr+27r^2=0$

$9pqr=2q^3+27r^2$

EX 3.22

It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progression. Find its roots.

Soln:

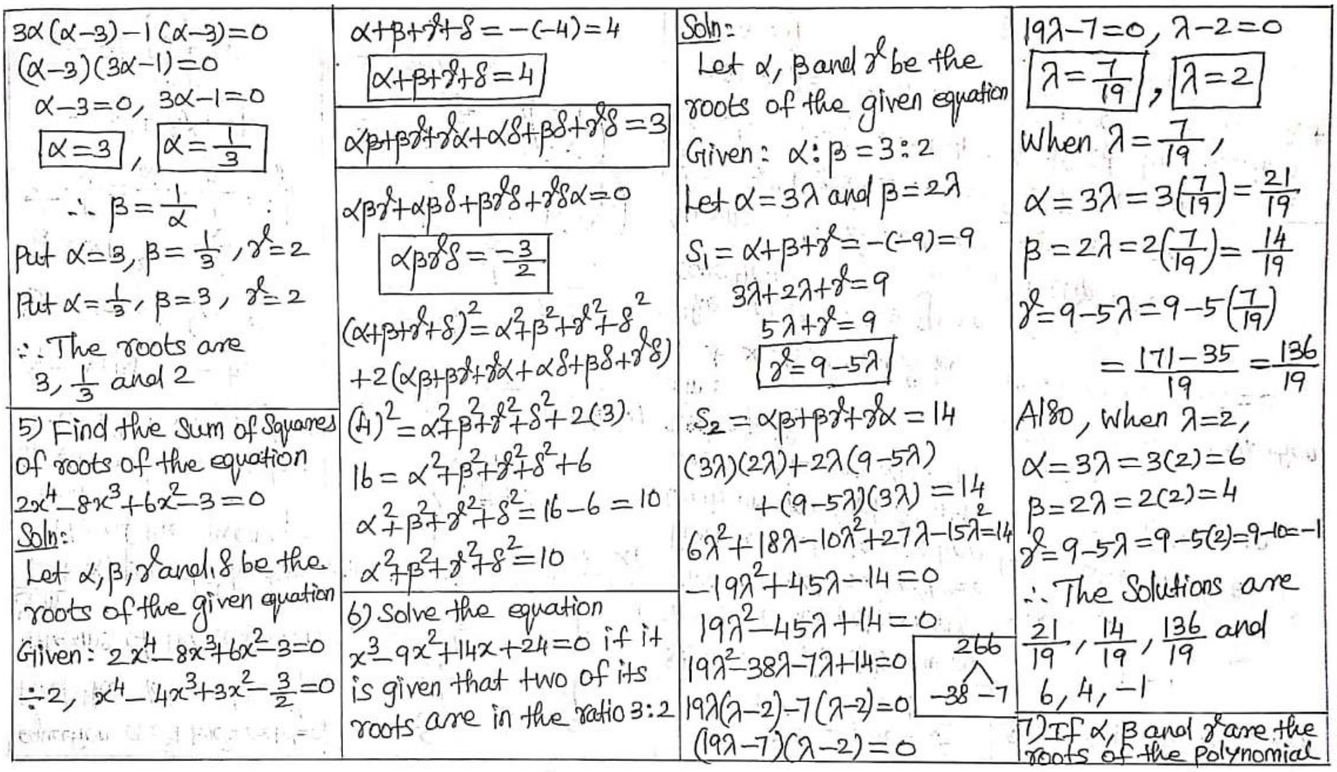
Let $\alpha-d,\alpha,\alpha+d$ be the roots in A.P

Given: $x^3-6x^2-4x+24=0$

$\alpha-\cancel{d}+\alpha+\alpha+\cancel{d}=-(-6)$

$\cancel{3}\alpha=\cancel{6}2$

$\boxed{\alpha=2}$



$3\alpha(\alpha-3)-1(\alpha-3)=0$
$(\alpha-3)(3\alpha-1)=0$
$\alpha-3=0,\ 3\alpha-1=0$
$\boxed{\alpha=3},\ \boxed{\alpha=\frac{1}{3}}$
$\therefore \beta=\frac{1}{\alpha}$
Put $\alpha=3,\ \beta=\frac{1}{3},\ \gamma=2$
Put $\alpha=\frac{1}{3},\ \beta=3,\ \gamma=2$
∴ The roots are
$3, \frac{1}{3}$ and 2

5) Find the sum of Squares of roots of the equation $2x^4-8x^3+6x^2-3=0$

Soln:
Let α,β,γ and δ be the roots of the given equation
Given: $2x^4-8x^3+6x^2-3=0$
$\div 2,\ x^4-4x^3+3x^2-\frac{3}{2}=0$

$\alpha+\beta+\gamma+\delta=-(-4)=4$
$\boxed{\alpha+\beta+\gamma+\delta=4}$
$\boxed{\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta=3}$
$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\gamma\delta\alpha=0$
$\boxed{\alpha\beta\gamma\delta=\frac{-3}{2}}$
$(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2$
$+2(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta)$
$(4)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2(3)$
$16=\alpha^2+\beta^2+\gamma^2+\delta^2+6$
$\alpha^2+\beta^2+\gamma^2+\delta^2=16-6=10$
$\alpha^2+\beta^2+\gamma^2+\delta^2=10$

6) Solve the equation $x^3-9x^2+14x+24=0$ if it is given that two of its roots are in the ratio 3:2

Soln:
Let α, β and γ be the roots of the given equation
Given: $\alpha:\beta=3:2$
Let $\alpha=3\lambda$ and $\beta=2\lambda$
$S_1=\alpha+\beta+\gamma=-(-9)=9$
$3\lambda+2\lambda+\gamma=9$
$5\lambda+\gamma=9$
$\boxed{\gamma=9-5\lambda}$
$S_2=\alpha\beta+\beta\gamma+\gamma\alpha=14$
$(3\lambda)(2\lambda)+2\lambda(9-5\lambda)$
$+(9-5\lambda)(3\lambda)=14$
$6\lambda^2+18\lambda-10\lambda^2+27\lambda-15\lambda^2=14$
$-19\lambda^2+45\lambda-14=0$
$19\lambda^2-45\lambda+14=0$
$19\lambda^2-38\lambda-7\lambda+14=0$
$19\lambda(\lambda-2)-7(\lambda-2)=0$
$(19\lambda-7)(\lambda-2)=0$

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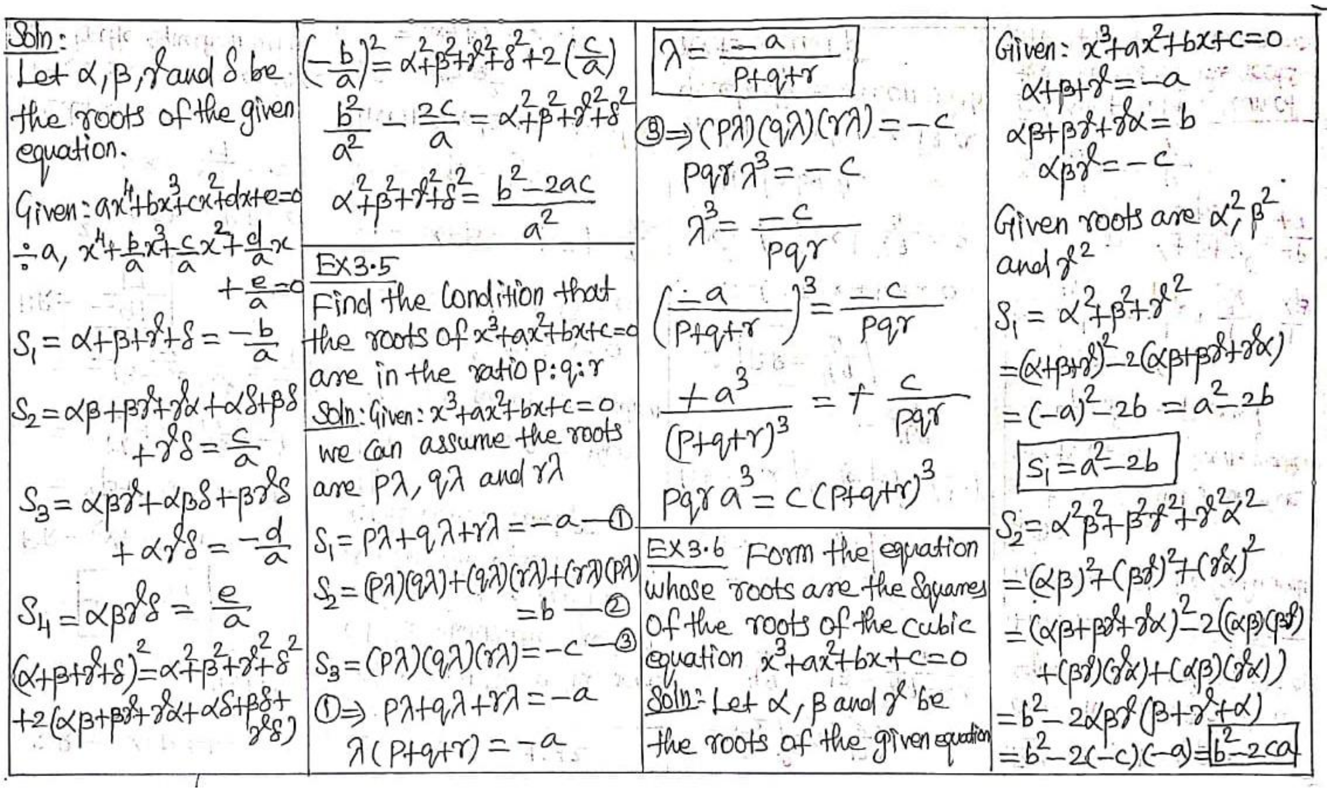
$-38\quad -7$

$19\lambda-7=0,\ \lambda-2=0$
$\boxed{\lambda=\frac{7}{19}},\ \boxed{\lambda=2}$
When $\lambda=\frac{7}{19}$,
$\alpha=3\lambda=3\left(\frac{7}{19}\right)=\frac{21}{19}$
$\beta=2\lambda=2\left(\frac{7}{19}\right)=\frac{14}{19}$
$\gamma=9-5\lambda=9-5\left(\frac{7}{19}\right)$
$=\frac{171-35}{19}=\frac{136}{19}$
Also, when $\lambda=2$,
$\alpha=3\lambda=3(2)=6$
$\beta=2\lambda=2(2)=4$
$\gamma=9-5\lambda=9-5(2)=9-10=-1$
∴ The Solutions are
$\frac{21}{19}, \frac{14}{19}, \frac{136}{19}$ and
$6, 4, -1$

7) If α, β and γ are the roots of the polynomial



Soln:
Let α,β,γ and δ be the roots of the given equation.
Given: $ax^4+bx^3+cx^2+dx+e=0$
$\div a,\ x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$
$S_1=\alpha+\beta+\gamma+\delta=-\frac{b}{a}$
$S_2=\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta=\frac{c}{a}$
$S_3=\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta=-\frac{d}{a}$
$S_4=\alpha\beta\gamma\delta=\frac{e}{a}$
$(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2$
$+2(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta)$

$\left(-\frac{b}{a}\right)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2\left(\frac{c}{a}\right)$
$\frac{b^2}{a^2}-\frac{2c}{a}=\alpha^2+\beta^2+\gamma^2+\delta^2$
$\alpha^2+\beta^2+\gamma^2+\delta^2=\frac{b^2-2ac}{a^2}$

EX 3.5
Find the condition that the roots of $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$

Soln: Given: $x^3+ax^2+bx+c=0$
we can assume the roots are $p\lambda$, $q\lambda$ and $r\lambda$
$S_1=p\lambda+q\lambda+r\lambda=-a$ — ①
$S_2=(p\lambda)(q\lambda)+(q\lambda)(r\lambda)+(r\lambda)(p\lambda)=b$ — ②
$S_3=(p\lambda)(q\lambda)(r\lambda)=-c$ — ③
① ⇒ $p\lambda+q\lambda+r\lambda=-a$
$\lambda(p+q+r)=-a$
$\boxed{\lambda=\frac{-a}{p+q+r}}$
③ ⇒ $(p\lambda)(q\lambda)(r\lambda)=-c$
$pqr\lambda^3=-c$
$\lambda^3=\frac{-c}{pqr}$
$\left(\frac{-a}{p+q+r}\right)^3=\frac{-c}{pqr}$
$\frac{\cancel{-}a^3}{(p+q+r)^3}=\cancel{-}\frac{c}{pqr}$
$pqra^3=c(p+q+r)^3$

EX 3.6 Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c=0$

Soln: Let α, β and γ be the roots of the given equation
Given: $x^3+ax^2+bx+c=0$
$\alpha+\beta+\gamma=-a$
$\alpha\beta+\beta\gamma+\gamma\alpha=b$
$\alpha\beta\gamma=-c$
Given roots are α^2, β^2 and γ^2
$S_1=\alpha^2+\beta^2+\gamma^2$
$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$=(-a)^2-2b=a^2-2b$
$\boxed{S_1=a^2-2b}$
$S_2=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$
$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2((\alpha\beta)(\beta\gamma)$
$+(\beta\gamma)(\gamma\alpha)+(\alpha\beta)(\gamma\alpha))$
$=b^2-2\alpha\beta\gamma(\beta+\gamma+\alpha)$
$=b^2-2(-c)(-a)=\boxed{b^2-2ca}$

$\therefore$ The other two roots are $\frac{1+\sqrt{37}}{2}$ and $\frac{1-\sqrt{37}}{2}$

$\therefore 1+2i, 1-2i, \sqrt{3}, -\sqrt{3},$ $\frac{1+\sqrt{37}}{2}$ and $\frac{1-\sqrt{37}}{2}$ are the roots of the given equation

6) Solve the cubic equations

(i) $2x^3-9x^2+10x=3$

(ii) $8x^3-2x^2-7x+3=0$

Soln: (i) Given:

$2x^3-9x^2+10x-3=0$

Sum of the coefficients $= 2-9+10-3=0$

$\therefore x=1$ is a root

$\therefore x-1$ is a factor

$$\begin{array}{c|cccc} 1 & 2 & -9 & 10 & -3 \\ & 0 & 2 & -7 & 3 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

[OR]

$$\begin{array}{r|l} & 2x^2-7x+3 \\ \hline x-1 & 2x^3-9x^2+10x-3 \\ & \cancel{2x^3} \mp 2x^2 \\ \hline & -7x^2+10x \\ & \cancel{-7x^2} \pm 7x \\ \hline & 3x-3 \\ & \cancel{3x} \mp 3 \\ \hline & 0 \end{array}$$

$(x-1)(2x^2-7x+3)$ is a factor.

$(x-1)(2x^2-7x+3)=0$

$x-1=0,\ 2x^2-7x+3=0$

$\boxed{x=1},\ 2x^2-6x-x+3=0$ $\quad \overset{6}{\wedge}_{-6\ -1}$

$2x(x-3)-1(x-3)=0$

$(2x-1)(x-3)=0$

$2x-1=0,\ x-3=0$

$\boxed{x=\frac{1}{2}},\ \boxed{x=3}$

$\therefore 1, \frac{1}{2}$ and 3 are the roots of the given Polynomial equation.

(ii) Given:

$8x^3-2x^2-7x+3=0$

Sum of the coefficients of odd power = sum of the coefficients of even power

$8-7=-2+3$

$1=1$

$$\begin{array}{c|cccc} -1 & 8 & -2 & -7 & 3 \\ & 0 & -8 & 10 & -3 \\ \hline & 8 & -10 & 3 & \boxed{0} \end{array}$$

$\therefore x=-1$ is a root

$\therefore x+1$ is a factor

[OR]

$$\begin{array}{r|l} & 8x^2-10x+3 \\ \hline x+1 & 8x^3-2x^2-7x+3 \\ & \cancel{8x^3} \pm 8x^2 \\ \hline & -10x^2-7x \\ & \cancel{-10x^2} \mp 10x \\ \hline & 3x+3 \\ & \cancel{3x} + \cancel{3} \\ \hline & 0 \end{array}$$

$(x+1)(8x^2-10x+3)$ is a factor.

$(x+1)(8x^2-10x+3)=0$

$x+1=0,\ 8x^2-10x+3=0$

$\boxed{x=-1},\ 8x^2-4x-6x+3=0$ $\quad \overset{24}{\wedge}_{-4\ -6}$

$4x(2x-1)-3(2x-1)=0$

$(4x-3)(2x-1)=0$

$4x-3=0,\ 2x-1=0$

$\boxed{x=\frac{3}{4}},\ \boxed{x=\frac{1}{2}}$

$\therefore -1, \frac{1}{2}$ and $\frac{3}{4}$ are the roots of the given Polynomial equation

7) Solve the equation $x^4-14x^2+45=0$

Soln: Given: $x^4-14x^2+45=0$

$(x^2)^2-14x^2+45=0$

Put $x^2=y$

$y^2-14y+45=0$ $\quad \overset{45}{\wedge}_{-9\ -5}$

$(y-9)(y-5)=0$

$y-9=0,\ y-5=0$

$y=9,\ y=5$

$S_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$

$= (-c)^2 = c^2$

$\boxed{S_3=c^2}$

$\therefore x^3-S_1x^2+S_2x-S_3=0$

$x^3-(a^2-2b)x^2+(b^2-2ca)x-c^2=0$

Ex 3.7

If P is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$ in terms of P

Soln: Given:

$4x^2+4px+(p+2)=0$

$a=4,\ b=4p,\ c=p+2$

$\Delta=b^2-4ac$

$=(4p)^2-4(4)(p+2)$

$=16p^2-16(p+2)$

$=16(p^2-(p+2))$

$=16(p^2-p-2)$ $\quad \overset{-2}{\wedge}_{1\ -2}$

$\Delta=16((p+1)(p-2))$

$-\infty \quad -1 \quad 2 \quad \infty$

if $-1<p<2 \Rightarrow \Delta<0$, imaginary roots

if $p=-1$ (or) $p=2 \Rightarrow \Delta=0$, equal real roots

if $-\infty<p<-1$ (or) $2<p<\infty$ $\Rightarrow \Delta>0$, distinct real roots

3.4 Nature of Roots and Nature of Coefficients of Polynomial Equations

3.4.1 Imaginary Roots

EXERCISE 3.2

1) If k is real, discuss the nature of the roots of the polynomial equation $2x^2+kx+k=0$, in terms of k

Soln: Given: $2x^2+kx+k=0$

$a=2,\ b=k,\ c=k$

$\Delta=b^2-4ac=k^2-4(2)k$

$=k^2-8k$

$\Delta=k(k-8)$

Put $\Delta=0$, $k(k-8)=0$

$\boxed{k=0},\ k-8=0$

$\boxed{k=8}$

$\xleftrightarrow{\quad} \quad -\infty \quad 0 \quad 8 \quad \infty$

When $k<0$, $\Delta>0$

The roots are real and distinct

When $k=0$ (or) $k=8$, $\Delta=0$

The roots are real and equal

When $0<k<8$, $\Delta<0$

The roots are imaginary

When $k>8$, $\Delta>0$

The roots are real and distinct

2) Find a polynomial equation of minimum degree with rational coefficients, having $2+\sqrt{3}i$ as a root

Soln:

Since $2+\sqrt{3}i$ is a root with rational coefficients $2-\sqrt{3}i$ is also a root

$S.R = 2+\cancel{\sqrt{3}i}+2-\cancel{\sqrt{3}i}=4$

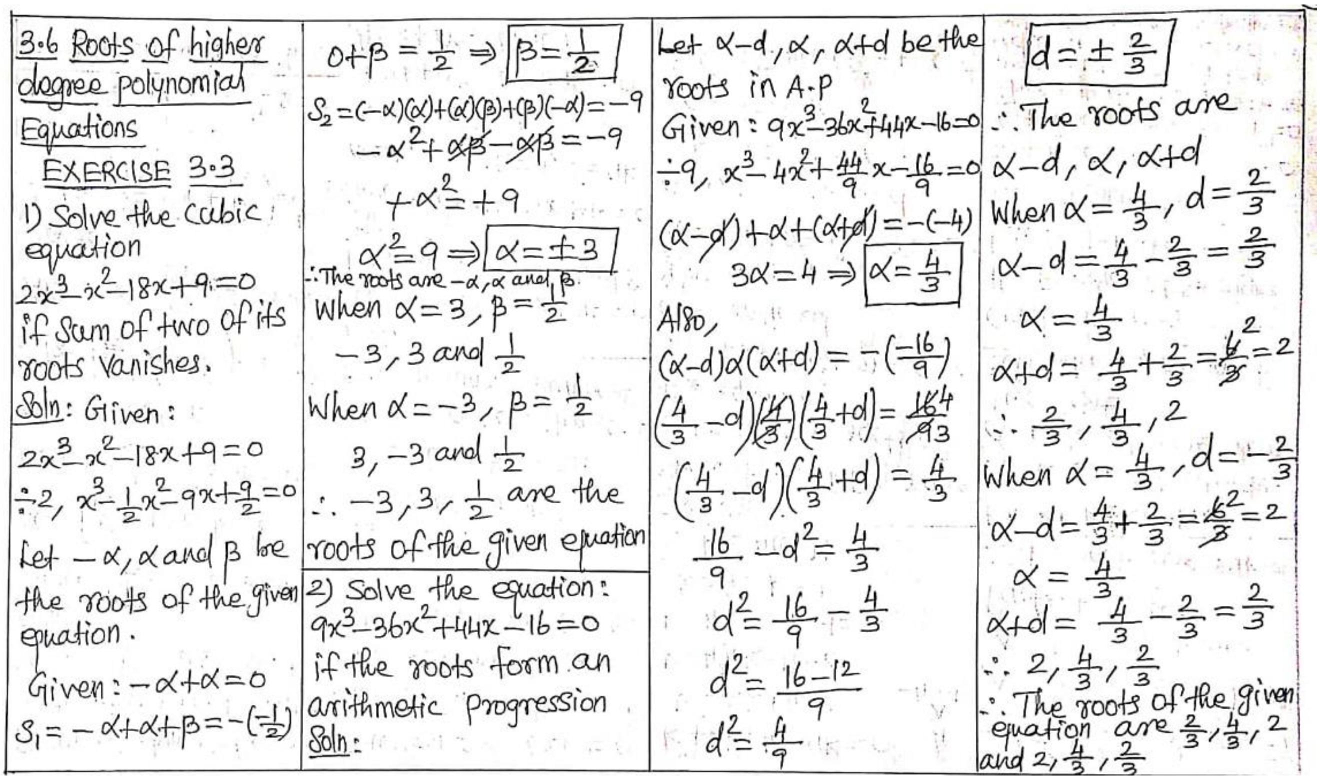
$P.R=(2+\sqrt{3}i)(2-\sqrt{3}i)=4+3=7$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-4x+7=0$

3) Find a polynomial equation of minimum degree with rational coefficients having $2i+3$ as a root

Soln: Since $3+2i$ is a root with rational coefficients $3-2i$ is also a root



3.6 Roots of higher degree polynomial Equations

EXERCISE 3.3

1) Solve the cubic equation $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.

Soln: Given:

$2x^3-x^2-18x+9=0$

$\div 2$, $x^3-\frac{1}{2}x^2-9x+\frac{9}{2}=0$

Let $-\alpha, \alpha$ and β be the roots of the given equation.

Given: $-\alpha+\alpha=0$

$S_1=-\alpha+\alpha+\beta=-\left(\frac{-1}{2}\right)$

$0+\beta=\frac{1}{2} \Rightarrow \boxed{\beta=\frac{1}{2}}$

$S_2=(-\alpha)(\alpha)+(\alpha)(\beta)+(\beta)(-\alpha)=-9$

$-\alpha^2+\cancel{\alpha\beta}-\cancel{\alpha\beta}=-9$

$+\alpha^2=+9$

$\alpha^2=9 \Rightarrow \boxed{\alpha=\pm 3}$

$\therefore$ The roots are $-\alpha, \alpha$ and β.

When $\alpha=3, \beta=\frac{1}{2}$

$-3, 3$ and $\frac{1}{2}$

When $\alpha=-3, \beta=\frac{1}{2}$

$3, -3$ and $\frac{1}{2}$

$\therefore -3, 3, \frac{1}{2}$ are the roots of the given equation

2) Solve the equation: $9x^3-36x^2+44x-16=0$ if the roots form an arithmetic progression

Soln:

Let $\alpha-d, \alpha, \alpha+d$ be the roots in A.P

Given: $9x^3-36x^2+44x-16=0$

$\div 9$, $x^3-4x^2+\frac{44}{9}x-\frac{16}{9}=0$

$(\alpha-\cancel{d})+\alpha+(\alpha+\cancel{d})=-(-4)$

$3\alpha=4 \Rightarrow \boxed{\alpha=\frac{4}{3}}$

Also,

$(\alpha-d)\alpha(\alpha+d)=-\left(\frac{-16}{9}\right)$

$\left(\frac{4}{3}-d\right)\cancel{\left(\frac{4}{3}\right)}\left(\frac{4}{3}+d\right)=\frac{\cancel{16}4}{\cancel{9}3}$

$\left(\frac{4}{3}-d\right)\left(\frac{4}{3}+d\right)=\frac{4}{3}$

$\frac{16}{9}-d^2=\frac{4}{3}$

$d^2=\frac{16}{9}-\frac{4}{3}$

$d^2=\frac{16-12}{9}$

$d^2=\frac{4}{9}$

$\boxed{d=\pm\frac{2}{3}}$

$\therefore$ The roots are

$\alpha-d, \alpha, \alpha+d$

When $\alpha=\frac{4}{3}, d=\frac{2}{3}$

$\alpha-d=\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$

$\alpha=\frac{4}{3}$

$\alpha+d=\frac{4}{3}+\frac{2}{3}=\frac{\cancel{6}^{2}}{\cancel{3}}=2$

$\therefore \frac{2}{3}, \frac{4}{3}, 2$

When $\alpha=\frac{4}{3}, d=-\frac{2}{3}$

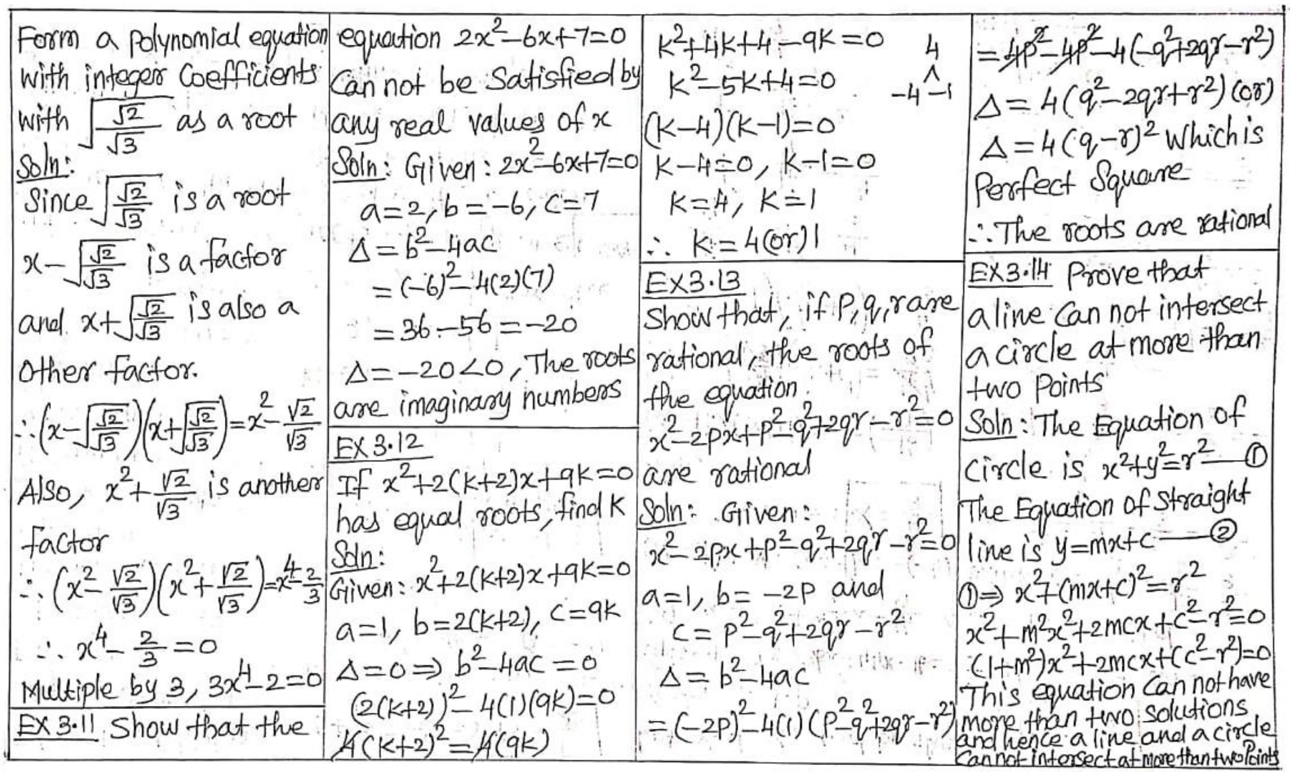
$\alpha-d=\frac{4}{3}+\frac{2}{3}=\frac{\cancel{6}^{2}}{\cancel{3}}=2$

$\alpha=\frac{4}{3}$

$\alpha+d=\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$

$\therefore 2, \frac{4}{3}, \frac{2}{3}$

$\therefore$ The roots of the given equation are $\frac{2}{3}, \frac{4}{3}, 2$ and $2, \frac{4}{3}, \frac{2}{3}$



Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root

Soln:

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root

$x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a factor

and $x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is also a other factor.

$\therefore \left(x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)=x^2-\frac{\sqrt{2}}{\sqrt{3}}$

Also, $x^2+\frac{\sqrt{2}}{\sqrt{3}}$ is another factor

$\therefore \left(x^2-\frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2+\frac{\sqrt{2}}{\sqrt{3}}\right)=x^4-\frac{2}{3}$

$\therefore x^4-\frac{2}{3}=0$

Multiple by 3, $3x^4-2=0$

EX 3.11 Show that the equation $2x^2-6x+7=0$ can not be satisfied by any real values of x

Soln: Given: $2x^2-6x+7=0$

$a=2, b=-6, c=7$

$\Delta=b^2-4ac$

$=(-6)^2-4(2)(7)$

$=36-56=-20$

$\Delta=-20<0$, The roots are imaginary numbers

EX 3.12

If $x^2+2(k+2)x+9k=0$ has equal roots, find K

Soln:

Given: $x^2+2(k+2)x+9k=0$

$a=1, b=2(k+2), c=9k$

$\Delta=0 \Rightarrow b^2-4ac=0$

$(2(k+2))^2-4(1)(9k)=0$

$\cancel{4}(k+2)^2=\cancel{4}(9k)$

$k^2+4k+4-9k=0$

$k^2-5k+4=0$

$$\overset{4}{\wedge}$$
$-4 \quad -1$

$(k-4)(k-1)=0$

$k-4=0, k-1=0$

$k=4, k=1$

$\therefore k=4$ (or) 1

EX 3.13

Show that, if p, q, r are rational, the roots of the equation $x^2-2px+p^2-q^2+2qr-r^2=0$ are rational

Soln: Given:

$x^2-2px+p^2-q^2+2qr-r^2=0$

$a=1, b=-2p$ and

$c=p^2-q^2+2qr-r^2$

$\Delta=b^2-4ac$

$=(-2p)^2-4(1)(p^2-q^2+2qr-r^2)$

$=\cancel{4p^2}-\cancel{4p^2}-4(-q^2+2qr-r^2)$

$\Delta=4(q^2-2qr+r^2)$ (or)

$\Delta=4(q-r)^2$ which is Perfect Square

$\therefore$ The roots are rational

EX 3.14 Prove that a line can not intersect a circle at more than two points

Soln: The Equation of circle is $x^2+y^2=r^2$ —— ①

The Equation of straight line is $y=mx+c$ —— ②

① $\Rightarrow x^2+(mx+c)^2=r^2$

$x^2+m^2x^2+2mcx+c^2-r^2=0$

$(1+m^2)x^2+2mcx+(c^2-r^2)=0$

This equation can not have more than two solutions and hence a line and a circle cannot intersect at more than two points

$S.R = 3+\cancel{2i}+3-\cancel{2i} = 6$

$P.R = (3+2i)(3-2i) = 9+4 = 13$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-6x+13=0$

4) Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root

Soln:

Since $\sqrt{5}-\sqrt{3}$ is a root with rational coefficients $\sqrt{5}+\sqrt{3}$ is also a root

$S.R = \sqrt{5}-\cancel{\sqrt{3}}+\sqrt{5}+\cancel{\sqrt{3}} = 2\sqrt{5}$

$P.R = (\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3}) = 5-3 = 2$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-2\sqrt{5}x+2=0$

Also,

Since $-\sqrt{5}-\sqrt{3}$ is a root with rational coefficients $-\sqrt{5}+\sqrt{3}$ is also a root

$S.R = -\sqrt{5}-\cancel{\sqrt{3}}-\sqrt{5}+\cancel{\sqrt{3}} = -2\sqrt{5}$

$P.R = (-\sqrt{5}-\sqrt{3})(-\sqrt{5}+\sqrt{3}) = 5-3$
$= 2$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-(-2\sqrt{5})x+2=0$

$x^2+2\sqrt{5}x+2=0$

$\therefore (x^2-2\sqrt{5}x+2)(x^2+2\sqrt{5}x+2)=0$

$((x^2+2)-2\sqrt{5})((x^2+2)+2\sqrt{5})=0$

$(x^2+2)^2-(2\sqrt{5})^2=0$

$x^4+4x^2+4-20x^2=0$

$x^4-16x^2+4=0$

5) Prove that a straight line and parabola cannot intersect at more than two points

Soln: The Equation of straight line is $y=mx+c$ —①

The Equation of Parabola is $y^2=4ax$ —②

Substitute eqn ① in ②

$(mx+c)^2=4ax$

$m^2x^2+2mcx+c^2-4ax=0$

$m^2x^2+(2mc-4a)x+c^2=0$

This equation cannot have more than two solutions and hence a straight line and parabola cannot intersect at more than two points.

EX 3.8 Find the monic Polynomial equation of minimum degree with real coefficients having $2-\sqrt{3}i$ as a root

Soln: since $2-\sqrt{3}i$ is a root with real coefficients $2+\sqrt{3}i$ is also a root

$S.R = 2-\cancel{\sqrt{3}i}+2+\cancel{\sqrt{3}i} = 4$

$P.R = (2-\sqrt{3}i)(2+\sqrt{3}i) = 4+3$
$= 7$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-4x+7=0$

EX 3.9

Find a Polynomial equation of minimum degree with rational Coefficients having $2-\sqrt{3}$ as a root

Soln:

Since $2-\sqrt{3}$ is a root with rational coefficients $2+\sqrt{3}$ is also a root

$S.R = 2-\cancel{\sqrt{3}}+2+\cancel{\sqrt{3}} = 4$

$P.R = (2-\sqrt{3})(2+\sqrt{3}) = 4-3$
$= 1$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-4x+1=0$

EX 3.10

3) Solve the equation $3x^3-26x^2+52x-24=0$ if its roots form a geometric progression

Soln:

Let $\frac{\alpha}{\lambda}, \alpha, \alpha\lambda$ be the roots in G.P

Given: $3x^3-26x^2+52x-24=0$

$\div 3$, $x^3-\frac{26}{3}x^2+\frac{52}{3}x-8=0$

$\frac{\alpha}{\lambda}+\alpha+\alpha\lambda = -\left(\frac{-26}{3}\right)$

$\alpha\left(\frac{1}{\lambda}+1+\lambda\right) = \frac{26}{3}$ —①

$\left(\frac{\alpha}{\cancel{\lambda}}\right)(\alpha)(\alpha\cancel{\lambda}) = -(-8)$

$\alpha^3 = 8 \Rightarrow \alpha^3 = 2^3$

$\boxed{\alpha=2}$

① $\Rightarrow \cancel{2}\left(\frac{1}{\lambda}+1+\lambda\right) = \frac{\cancel{26}^{13}}{3}$

$\left(\frac{1+\lambda+\lambda^2}{\lambda}\right) = \frac{13}{3}$

$3(1+\lambda+\lambda^2) = 13\lambda$

$3+3\lambda+3\lambda^2-13\lambda=0$ $\quad 9$

$3\lambda^2-10\lambda+3=0$ $\quad \overset{\wedge}{-9\ -1}$

$3\lambda^2-9\lambda-\lambda+3=0$

$3\lambda(\lambda-3)-1(\lambda-3)=0$

$(3\lambda-1)(\lambda-3)=0$

$3\lambda-1=0,\ \lambda-3=0$

$\boxed{\lambda=\frac{1}{3}}, \boxed{\lambda=3}$

$\therefore \frac{\alpha}{\lambda}, \alpha, \alpha\lambda$ are roots in G.P

When $\alpha=2,\ \lambda=3$

$\frac{\alpha}{\lambda} = \frac{2}{3},\ \alpha=2,$

$\alpha\lambda = (2)(3) = 6$

$\therefore \frac{2}{3}, 2, 6$

When $\alpha=2,\ \lambda=\frac{1}{3}$

$\frac{\alpha}{\lambda} = \frac{2}{\frac{1}{3}} = 6$

$\alpha=2,\ \alpha\lambda = (2)\left(\frac{1}{3}\right) = \frac{2}{3}$

$\therefore 6, 2, \frac{2}{3}$

$\therefore$ The roots of the given equation are $\frac{2}{3}, 2, 6$ and $6, 2, \frac{2}{3}$

4) Determine k and Solve the equation $2x^3-6x^2+3x+k=0$ if one of its roots is twice the sum of the other two roots

Soln: Let $\alpha, \beta, 2(\alpha+\beta)$ be the roots of the given equation

$\alpha+\beta+2(\alpha+\beta) = -\left(\frac{-\cancel{6}^{3}}{\cancel{2}}\right)$

$(\alpha+\beta)+2(\alpha+\beta) = 3$

$\cancel{3}(\alpha+\beta) = \cancel{3}$

$\alpha+\beta = 1 \Rightarrow \boxed{\beta = 1-\alpha}$

$\therefore 2(\alpha+\beta) = 2(1) = 2$

$\therefore$ The roots are α, $1-\alpha$ and 2

$\therefore$ 2 is one of the root of the given eqn.

When $x=2$,

$2(2)^3-6(2)^2+3(2)+k=0$

$16-24+6+k=0$

$-2+k=0$

$\boxed{k=2}$

$\therefore 2x^3-6x^2+3x+2=0$

$\div 2$, $x^3-3x^2+\frac{3}{2}x+1=0$

$x = \frac{-3\pm\sqrt{5}}{2}$

∴ The required Solutions of the given equation are $1, -1, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

6) Find all real numbers Satisfying $4^x - 3(2^{x+2}) + 2^5 = 0$

Soln: Given:

$4^x - 3(2^{x+2}) + 2^5 = 0$

$(2^2)^x - 3(2^x 2^2) + 32 = 0$

$2^{2x} - 3(2^x)(4) + 32 = 0$

$(2^x)^2 - 12(2^x) + 32 = 0$

Take $2^x = y$

$y^2 - 12y + 32 = 0$ $\quad \overset{32}{\wedge}\ -8\ -4$

$(y-8)(y-4) = 0$

$y-8=0, y-4=0$

$y=8, y=4$

Put $y = 2^x$

$2^x = 8$, $2^x = 4$

$2^x = 2^3$, $2^x = 2^2$

∴ $x=3$, $x=2$

∴ The real numbers are 2, 3

7) Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a Solution

Soln: Given:

$6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$

[Let $P(x) = 6x^4 - 5x^3 - 38x^2 - 5x + 6$.

Also,

$P(\frac{1}{x}) = 6(\frac{1}{x})^4 - 5(\frac{1}{x})^3 - 38(\frac{1}{x})^2 - 5(\frac{1}{x}) + 6$

$= 6(\frac{1}{x^4}) - 5(\frac{1}{x^3}) - 38(\frac{1}{x^2}) - \frac{5}{x} + 6$

$= \frac{6 - 5x - 38x^2 - 5x^3 + 6x^4}{x^4}$

$= \frac{0}{x^4} = 0 \Rightarrow P(\frac{1}{x}) = 0$]

From the reciprocal equation,

If α is a root then $\frac{1}{\alpha}$ is also a root

Given: $\frac{1}{3}$ is a root and 3 is also a root

∴ $\boxed{x = \frac{1}{3}}$ is a root

$3x-1$ is a factor

Also, $\boxed{x=3}$ is a root

$x-3$ is a factor

∴ $(3x-1)(x-3)$

$= 3x^2 - 10x + 3$ is a factor

$$\begin{array}{r|l} & 2x^2+5x+2 \\ \hline 3x^2-10x+3 & 6x^4-5x^3-38x^2-5x+6 \\ & \cancel{6x^4} \underset{+}{-}20x^3 \underset{-}{+} 6x^2 \\ \hline & \cancel{15x^3}-44x^2-5x \\ & \underset{-}{}\cancel{15x^3}\underset{+}{-}50x^2\underset{-}{+}15x \\ \hline & \cancel{6x^2}-\cancel{20x}+\cancel{6} \\ & \underset{-}{}\cancel{6x^2}\underset{+}{-}\cancel{20x}\underset{-}{+}\cancel{6} \\ \hline & 0 \end{array}$$

∴ $2x^2+5x+2$ is also a factor

∴ $2x^2+5x+2 = 0$ $\quad \overset{4}{\wedge}\ 4\ 1$

$2x^2+4x+x+2=0$

$2x(x+2)+1(x+2)=0$

$(2x+1)(x+2)=0$

$2x+1=0$, $x+2=0$

$\boxed{x = \frac{-1}{2}}$, $\boxed{x=-2}$

∴ The Solution of the given equation are $\frac{1}{3}, 3, -\frac{1}{2}, -2$

Put $y = x^2$,

$x^2 = 9$, $x^2 = 5$

$\boxed{x = \pm 3}$, $\boxed{x = \pm\sqrt{5}}$

∴ The roots of the given polynomial equation are $3, -3, \sqrt{5}$ and $-\sqrt{5}$

EX 3.15

If $2+i$ and $3-\sqrt{2}$ are roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$, find all roots

Soln:

Since $2+i$ and $3-\sqrt{2}$ are roots, $2-i$ and $3+\sqrt{2}$ are also roots

$(x-(2+i)), (x-(2-i)), (x-(3-\sqrt{2}))$ and $(x-(3+\sqrt{2}))$ are factors

$(x-(2+i))(x-(2-i))(x-(3-\sqrt{2})) \times (x-(3+\sqrt{2}))$

$= ((x-2)-i)((x-2)+i)((x-3)+\sqrt{2})((x-3)-\sqrt{2})$

$= ((x-2)^2+1)((x-3)^2-2)$

$= (x^2-4x+4+1)(x^2-6x+9-2)$

$= (x^2-4x+5)(x^2-6x+7)$

$= x^4 - 6x^3 + 7x^2 - 4x^3 + 24x^2 - 28x + 5x^2 - 30x + 35$

$= x^4 - 10x^3 + 36x^2 - 58x + 35$

$$\begin{array}{r|l} & x^2-3x-4 \\ \hline x^4-10x^3+36x^2-58x+35 & x^6-13x^5+62x^4-126x^3+65x^2+127x-140 \\ & \cancel{x^6}\underset{+}{-}10x^5\underset{-}{+}36x^4\underset{+}{-}58x^3\underset{-}{+}35x^2 \\ \hline & -\cancel{3x^5}+26x^4-68x^3+30x^2+127x \\ & \underset{+}{-}\cancel{3x^5}\underset{-}{+}30x^4\underset{+}{-}108x^3\underset{-}{+}174x^2\underset{+}{-}105x \\ \hline & -\cancel{4x^4}+\cancel{40x^3}-\cancel{144x^2}+\cancel{232x}+\cancel{140} \\ & \underset{+}{-}\cancel{4x^4}\underset{-}{+}\cancel{40x^3}\underset{+}{-}\cancel{144x^2}\underset{-}{+}\cancel{232x}\underset{-}{+}\cancel{140} \\ \hline & 0 \end{array}$$

∴ The other factor is x^2-3x-4

∴ $x^2-3x-4=0$ $\quad \overset{-4}{\wedge}\ -4\ 1$

$(x-4)(x+1)=0$

$x-4=0$, $x+1=0$

$\boxed{x=4}$, $\boxed{x=-1}$

∴ The other two roots are 4 and −1

∴ $2+i, 2-i, 3-\sqrt{2}, 3+\sqrt{2}$, 4 and −1 are the roots of the given polynomial equation.

EX 3.16 Solve the equation $x^4 - 9x^2 + 20 = 0$

Soln: Given: $x^4 - 9x^2 + 20 = 0$

$(x^2)^2 - 9x^2 + 20 = 0$

Put $x^2 = y$, $y^2 - 9y + 20 = 0$

$(y-4)(y-5)=0$ $\quad \overset{20}{\wedge}\ -4\ -5$

$y-4=0$ and $y-5=0$

$y=4$ and $y=5$

Put $y = x^2$,

$x^2 = 4$ and $x^2 = 5$

$x = \pm 2$ and $x = \pm\sqrt{5}$

∴ $2, -2, \sqrt{5}$ and $-\sqrt{5}$ are solutions of the given equation.

EX 3.17

Solve the equation $x^3 - 3x^2 - 33x + 35 = 0$

Soln: Given:

$x^3 - 3x^2 - 33x + 35 = 0$

Sum of Coefficients $= 1 - 3 - 33 + 35 = 0$

∴ $x=1$ is a root

∴ $x-1$ is a factor

$S.R = \alpha+\beta+4 = \frac{-43}{17}+4$

$= \frac{-43+68}{17} = \frac{25}{17}$

$\therefore \boxed{S.R = \frac{25}{17}}$

$P.R = (\alpha+2)(\beta+2)$

$= \alpha\beta+2(\alpha+\beta)+4$

$= \frac{-73}{17}+2\left(\frac{-43}{17}\right)+4$

$= \frac{-73-86+68}{17} = \frac{-91}{17}$

$\boxed{P.R = \frac{-91}{17}}$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-\frac{25}{17}x-\frac{91}{17}=0$

Multiple by 17,

$17x^2-25x-91=0$ is a quadratic equation with roots $\alpha+2$ and $\beta+2$

EX 3.2

If α and β are the roots of the quadratic equation $2x^2-7x+13=0$ Construct a quadratic equation whose roots are α^2 and β^2

Soln:

Let α and β are the roots of the given equation

Given: $2x^2-7x+13=0$

$\div 2$, $x^2-\frac{7}{2}x+\frac{13}{2}=0$

$S_1 = \alpha+\beta = -\left(\frac{-7}{2}\right) = \frac{7}{2}$

$S_2 = \alpha\beta = \frac{13}{2}$

Given roots are α^2 and β^2

$S.R = \alpha^2+\beta^2$

$= (\alpha+\beta)^2-2\alpha\beta$

$= \left(\frac{7}{2}\right)^2-\cancel{2}\left(\frac{13}{\cancel{2}}\right)$

$= \frac{49}{4}-13 = \frac{49-52}{4}$

$= \frac{-3}{4}$

$\boxed{S.R = -\frac{3}{4}}$

$P.R = \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{13}{2}\right)^2$

$= \frac{169}{4}$

$\boxed{P.R = \frac{169}{4}}$

$\therefore x^2-(S.R)x+P.R=0$

$x^2-\left(\frac{-3}{4}\right)x+\frac{169}{4}=0$

Multiple by 4,

$4x^2+3x+169=0$ is a quadratic equation with roots α^2 and β^2

EX3.3 If α, β and γ are the roots of the equation $x^3+px^2+qx+r=0$, Find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the Coefficients

Soln:

Let α, β and γ be the roots of the given equation

$\alpha+\beta+\gamma = -p$ and

$\alpha\beta\gamma = -r$

$\sum\frac{1}{\beta\gamma} = \frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}$

$= \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{+p}{+r}$

$\sum\frac{1}{\beta\gamma} = \frac{p}{r}$

EX 3.4 Find the Sum of the Squares of the roots of $ax^4+bx^3+cx^2+dx+e=0$

$a=1, b=-5, c=-13$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-13)}}{2(1)}$

$= \frac{5\pm\sqrt{25+52}}{2}$

$x = \frac{5\pm\sqrt{77}}{2}$

and $x^2-5x+5=0$

$a=1, b=-5, c=5$

$x = \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(5)}}{2(1)}$

$= \frac{5\pm\sqrt{25-20}}{2}$

$x = \frac{5\pm\sqrt{5}}{2}$

$\therefore$ The roots of the given equation are $\frac{5\pm\sqrt{77}}{2}$ and $\frac{5\pm\sqrt{5}}{2}$

EX3.24 [Book Qn Wrong]

Solve the equation $(2x-3)(6x-1)(3x-2)(x-2)-5=0$

Soln: Given:

$(2x-3)(6x-1)(3x-2)(x-2)-5=0$

$(2x-3)(3x-2)(6x-1)(x-2)-5=0$

$(6x^2-13x+6)(6x^2-13x+2)-5=0$

Take $6x^2-13x=y$

$(y+6)(y+2)-5=0$

$y^2+8y+12-5=0$ $\quad \underset{7\quad 1}{\overset{7}{\wedge}}$

$y^2+8y+7=0$

$(y+7)(y+1)=0$

Put $y=6x^2-13x$

$(6x^2-13x+7)(6x^2-13x+1)=0$

$6x^2-13x+7=0$, $6x^2-13x+1=0$

Now, $\quad \underset{-6\quad -7}{\overset{42}{\wedge}}$

$6x^2-13x+7=0$

$6x^2-6x-7x+7=0$

$6x(x-1)-7(x-1)=0$

$(6x-7)(x-1)=0$

$6x-7=0$, $x-1=0$

$x=\frac{7}{6}$, $x=1$

and $6x^2-13x+1=0$

$a=6$, $b=-13$, $c=1$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-13)\pm\sqrt{(-13)^2-4(6)(1)}}{2(6)}$

$= \frac{13\pm\sqrt{169-24}}{12}$

$x = \frac{13\pm\sqrt{145}}{12}$

$\therefore$ The roots of the given equation are $x=1$, $x=\frac{7}{6}$, $x=\frac{13+\sqrt{145}}{12}$ and $x=\frac{13-\sqrt{145}}{12}$

3.8 Polynomial Equations with no additional information

3.8.1 Rational Root Theorem

EXERCISE 3.5

1) Solve the following equations:

(i) $\sin^2x-5\sin x+4=0$

(ii) $12x^3+8x=29x^2-4$

Soln: (i) Given:

$\sin^2x-5\sin x+4=0$

Take $\sin x=y$ $\quad \underset{-4\quad -1}{\overset{4}{\wedge}}$

$y^2-5y+4=0$

$(y-4)(y-1)=0$

$y-4=0$, $y-1=0$

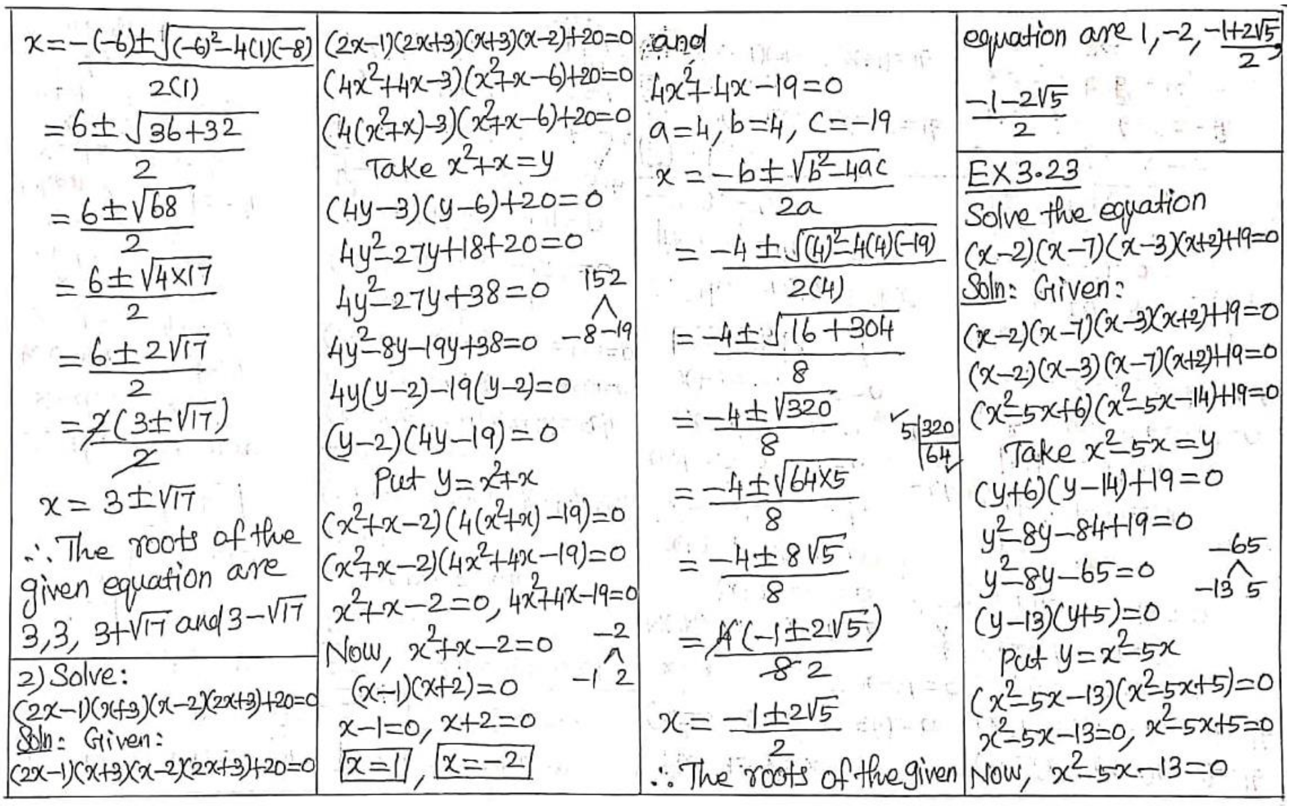
$y=4$, $y=1$

Put $y=\sin x$

$\sin x=4$ is not possible

$\sin x=1 \Rightarrow \sin x=\sin\frac{\pi}{2}$

$\therefore \sin x=1$, $x=n\pi+(-1)^n\frac{\pi}{2}$, $n\in Z$



$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-8)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36+32}}{2}$$
$$= \frac{6 \pm \sqrt{68}}{2}$$
$$= \frac{6 \pm \sqrt{4 \times 17}}{2}$$
$$= \frac{6 \pm 2\sqrt{17}}{2}$$
$$= \frac{\cancel{2}(3 \pm \sqrt{17})}{\cancel{2}}$$
$$x = 3 \pm \sqrt{17}$$

∴ The roots of the given equation are 3, 3, $3+\sqrt{17}$ and $3-\sqrt{17}$

2) Solve:
$(2x-1)(x+3)(x-2)(2x+3)+20=0$

Soln: Given:
$(2x-1)(x+3)(x-2)(2x+3)+20=0$
$(2x-1)(2x+3)(x+3)(x-2)+20=0$
$(4x^2+4x-3)(x^2+x-6)+20=0$
$(4(x^2+x)-3)(x^2+x-6)+20=0$

Take $x^2+x=y$
$(4y-3)(y-6)+20=0$
$4y^2-27y+18+20=0$
$4y^2-27y+38=0$ 152 $\wedge$ -8 -19
$4y^2-8y-19y+38=0$
$4y(y-2)-19(y-2)=0$
$(y-2)(4y-19)=0$

Put $y=x^2+x$
$(x^2+x-2)(4(x^2+x)-19)=0$
$(x^2+x-2)(4x^2+4x-19)=0$
$x^2+x-2=0,\ 4x^2+4x-19=0$
Now, $x^2+x-2=0$ -2 $\wedge$ -1 2
$(x-1)(x+2)=0$
$x-1=0,\ x+2=0$
$\boxed{x=1}, \boxed{x=-2}$

and
$4x^2+4x-19=0$
$a=4,\ b=4,\ c=-19$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{(4)^2-4(4)(-19)}}{2(4)}$$
$$= \frac{-4 \pm \sqrt{16+304}}{8}$$
$$= \frac{-4 \pm \sqrt{320}}{8}$$ 5 | 320, 64
$$= \frac{-4 \pm \sqrt{64 \times 5}}{8}$$
$$= \frac{-4 \pm 8\sqrt{5}}{8}$$
$$= \frac{\cancel{4}(-1 \pm 2\sqrt{5})}{\cancel{8}\,2}$$
$$x = \frac{-1 \pm 2\sqrt{5}}{2}$$

∴ The roots of the given equation are 1, −2, $\frac{-1+2\sqrt{5}}{2}$, $\frac{-1-2\sqrt{5}}{2}$

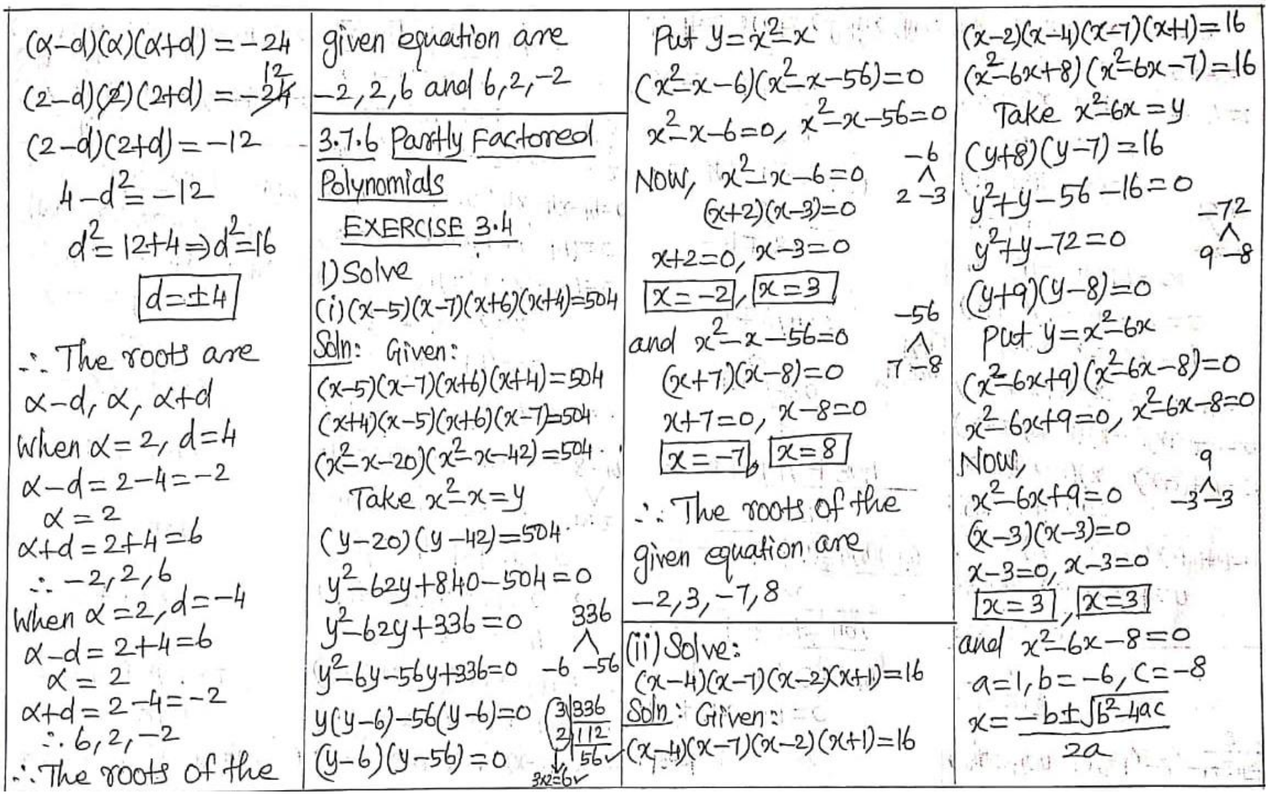
EX 3.23

Solve the equation
$(x-2)(x-7)(x-3)(x+2)+19=0$

Soln: Given:
$(x-2)(x-7)(x-3)(x+2)+19=0$
$(x-2)(x-3)(x-7)(x+2)+19=0$
$(x^2-5x+6)(x^2-5x-14)+19=0$

Take $x^2-5x=y$
$(y+6)(y-14)+19=0$
$y^2-8y-84+19=0$
$y^2-8y-65=0$ -65 $\wedge$ -13 5
$(y-13)(y+5)=0$

Put $y=x^2-5x$
$(x^2-5x-13)(x^2-5x+5)=0$
$x^2-5x-13=0,\ x^2-5x+5=0$
Now, $x^2-5x-13=0$



$(\alpha-d)(\alpha)(\alpha+d)=-24$
$(2-d)(\cancel{2})(2+d)=-\cancel{24}\,12$
$(2-d)(2+d)=-12$
$4-d^2=-12$
$d^2=12+4 \Rightarrow d^2=16$
$\boxed{d=\pm 4}$

∴ The roots are
$\alpha-d,\ \alpha,\ \alpha+d$
When $\alpha=2,\ d=4$
$\alpha-d=2-4=-2$
$\alpha=2$
$\alpha+d=2+4=6$
∴ −2, 2, 6
When $\alpha=2,\ d=-4$
$\alpha-d=2+4=6$
$\alpha=2$
$\alpha+d=2-4=-2$
∴ 6, 2, −2
∴ The roots of the given equation are −2, 2, 6 and 6, 2, −2

3.7.6 Partly Factored Polynomials

EXERCISE 3.4

1) Solve
(i) $(x-5)(x-7)(x+6)(x+4)=504$

Soln: Given:
$(x-5)(x-7)(x+6)(x+4)=504$
$(x+4)(x-5)(x+6)(x-7)=504$
$(x^2-x-20)(x^2-x-42)=504$

Take $x^2-x=y$
$(y-20)(y-42)=504$
$y^2-62y+840-504=0$
$y^2-62y+336=0$ 336 $\wedge$ -6 -56
$y^2-6y-56y+336=0$
$y(y-6)-56(y-6)=0$ 3 | 336, 2 | 112, 56 ✓ 3×2=6 ✓
$(y-6)(y-56)=0$

Put $y=x^2-x$
$(x^2-x-6)(x^2-x-56)=0$
$x^2-x-6=0,\ x^2-x-56=0$
Now, $x^2-x-6=0$ -6 $\wedge$ 2 -3
$(x+2)(x-3)=0$
$x+2=0,\ x-3=0$
$\boxed{x=-2}, \boxed{x=3}$
and $x^2-x-56=0$ -56 $\wedge$ 7 -8
$(x+7)(x-8)=0$
$x+7=0,\ x-8=0$
$\boxed{x=-7}, \boxed{x=8}$
∴ The roots of the given equation are −2, 3, −7, 8

(ii) Solve:
$(x-4)(x-7)(x-2)(x+1)=16$

Soln: Given:
$(x-4)(x-7)(x-2)(x+1)=16$
$(x-2)(x-4)(x-7)(x+1)=16$
$(x^2-6x+8)(x^2-6x-7)=16$

Take $x^2-6x=y$
$(y+8)(y-7)=16$
$y^2+y-56-16=0$
$y^2+y-72=0$ -72 $\wedge$ 9 -8
$(y+9)(y-8)=0$

Put $y=x^2-6x$
$(x^2-6x+9)(x^2-6x-8)=0$
$x^2-6x+9=0,\ x^2-6x-8=0$
Now,
$x^2-6x+9=0$ 9 $\wedge$ -3 -3
$(x-3)(x-3)=0$
$x-3=0,\ x-3=0$
$\boxed{x=3}, \boxed{x=3}$
and $x^2-6x-8=0$
$a=1,\ b=-6,\ c=-8$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x-1 \overline{) \; x^3-3x^2-33x+35 \;}\quad \text{quotient } x^2-2x-35$$
$x^3 - x^2$

$-2x^2-33x$

$-2x^2+2x$

$-35x+35$

$-35x+35$

0

$(x-1)(x^2-2x-35)$ is a factor

$(x-1)(x^2-2x-35)=0$

$x-1=0,\ x^2-2x-35=0$

$\boxed{x=1},\quad (x-7)(x+5)=0 \qquad \overset{-35}{\wedge}\ -7\ 5$

$x-7=0,\ x+5=0$

$\boxed{x=7},\ \boxed{x=-5}$

$\therefore$ 1, 7 and -5 are the roots of the given equation

Ex 3.18

Solve the equation $2x^3+11x^2-9x-18=0$

Soln:

Sum of the Coefficient of odd powers = Sum of the Coefficient of even powers

$2-9=11-18$

$-7=-7$

$\therefore x=-1$ is a root

$\therefore x+1$ is a factor

$$x+1 \overline{) \; 2x^3+11x^2-9x-18 \;}\quad \text{quotient } 2x^2+9x-18$$
$2x^3+2x^2$

$9x^2-9x$

$9x^2+9x$

$-18x-18$

$-18x-18$

0

$\therefore (x+1)(2x^2+9x-18)$ is a factor

$\therefore (x+1)(2x^2+9x-18)=0 \qquad \overset{-36}{\wedge}\ 12\ -3$

$x+1=0,\ 2x^2+9x-18=0$

$\boxed{x=-1},\ 2x^2+12x-3x-18=0$

$2x(x+6)-3(x+6)=0$

$(2x-3)(x+6)=0$

$2x-3=0,\ x+6=0$

$\boxed{x=\frac{3}{2}},\ \boxed{x=-6}$

$\therefore -6, -1, \frac{3}{2}$ are the roots of the given equation

Ex 3.19

obtain the Condition that the roots of $x^3+px^2+qx+r=0$ are in A.P

Soln:

Let $\alpha-d, \alpha, \alpha+d$ be the roots in A.P

Given: $x^3+px^2+qx+r=0$ — ①

$(\alpha-\cancel{d})+\alpha+(\alpha+\cancel{d})=-p$

$3\alpha=-p$

$\boxed{\alpha=\frac{-p}{3}}$

$\therefore \alpha$ is a root of the given equation.

$① \Rightarrow \left(\frac{-p}{3}\right)^3+p\left(\frac{-p}{3}\right)^2+q\left(\frac{-p}{3}\right)+r=0$

$\frac{-p^3}{27}+\frac{p^3}{9}-\frac{pq}{3}+r=0$

Multiple by 27,

$-p^3+3p^3-9pq+27r=0$

$2p^3-9pq+27r=0$

$9pq=2p^3+27r$

Ex 3.20

Find the Condition that the roots of $ax^3+bx^2+cx+d=0$ are in geometric progression. Assume $a,b,c,d \neq 0$

Soln: Let $\frac{\alpha}{\lambda}, \alpha, \alpha\lambda$ be the roots in G.P

Given: $ax^3+bx^2+cx+d=0$

$\frac{\alpha}{\lambda}+\alpha+\alpha\lambda=\frac{-b}{a}$

$\therefore$ The roots are $\alpha, 1-\alpha, 2$

$(\alpha)(1-\alpha)(2)=-1$

$2\alpha(1-\alpha)=-1$

$2\alpha-2\alpha^2=-1$

$2\alpha^2-2\alpha-1=0$

$a=2,\ b=-2,\ c=-1$

$\alpha=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-1)}}{2(2)}$

$=\frac{2\pm\sqrt{4+8}}{4}$

$=\frac{2\pm\sqrt{12}}{4}$

$=\frac{2\pm2\sqrt{3}}{4}$

$=\frac{\cancel{2}(1\pm\sqrt{3})}{\cancel{4}\,2}$

$\therefore \alpha=\frac{1\pm\sqrt{3}}{2}$

$\therefore \alpha=\frac{1+\sqrt{3}}{2}$ and $\alpha=\frac{1-\sqrt{3}}{2}$

The roots are $\alpha, 1-\alpha, 2$

When $\alpha=\frac{1+\sqrt{3}}{2}$,

$\alpha=\frac{1+\sqrt{3}}{2},\ 1-\alpha=1-\left(\frac{1+\sqrt{3}}{2}\right)$

$=\frac{2-1-\sqrt{3}}{2}$

$1-\alpha=\frac{1-\sqrt{3}}{2}$ and 2

When $\alpha=\frac{1-\sqrt{3}}{2}$,

$\alpha=\frac{1-\sqrt{3}}{2},\ 1-\alpha=1-\left(\frac{1-\sqrt{3}}{2}\right)$

$=\frac{2-1+\sqrt{3}}{2}$

$1-\alpha=\frac{1+\sqrt{3}}{2}$ and 2

$\therefore$ The roots are $\frac{1+\sqrt{3}}{2}$, $\frac{1-\sqrt{3}}{2}$, 2 and the value of k is 2

5) Find all zeros of the Polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that $1+2i$ and $\sqrt{3}$ are two of its zeros

Soln:

Since $1+2i$ and $\sqrt{3}$ are roots, $1-2i$ and $-\sqrt{3}$ are also roots.

$(x-(1+2i)), (x-(1-2i)), (x-\sqrt{3})$ and $(x+\sqrt{3})$ are factors.

$(x-(1+2i))(x-(1-2i))(x-\sqrt{3})(x+\sqrt{3})=0$

$((x-1)-2i)((x-1)+2i)(x-\sqrt{3})(x+\sqrt{3})=0$

$((x-1)^2+4)(x^2-3)=0$

$(x^2-2x+1+4)(x^2-3)=0$

$(x^2-2x+5)(x^2-3)=0$

$x^4-3x^2-2x^3+6x+5x^2-15=0$

$x^4-2x^3+2x^2+6x-15=0$

$$x^4-2x^3+2x^2+6x-15 \overline{) \; x^6-3x^5-5x^4+22x^3-39x^2-39x+135 \;}\quad \text{quotient } x^2-x-9$$
$x^6-2x^5+2x^4+6x^3-15x^2$

$-x^5-7x^4+16x^3-24x^2-39x$

$-x^5+2x^4-2x^3-6x^2+15x$

$-9x^4+18x^3-18x^2-54x+135$

$-9x^4+18x^3-18x^2-54x+135$

0

The other factor is x^2-x-9

$\therefore x^2-x-9=0$

$a=1,\ b=-1,\ c=-9$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-9)}}{2(1)}$

$=\frac{1\pm\sqrt{1+36}}{2}$

$x=\frac{1\pm\sqrt{37}}{2}$

$\therefore x = -\frac{1}{2}$ is not a root

Now, $x = 1$ is a root

$\therefore x-1$ is a factor

$$\begin{array}{c|cccc} 1 & 2 & -1 & 0 & -1 \\ & 0 & 2 & 1 & 1 \\ \hline & 2 & 1 & 1 & \boxed{0} \end{array}$$

$\therefore (x-1)(2x^2+x+1)$ is a factor

$(x-1)(2x^2+x+1) = 0$

$x-1=0, \quad 2x^2+x+1=0$

$\boxed{x=1}, \quad a=2, b=1, c=1$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1\pm\sqrt{(1)^2-4(2)(1)}}{2(2)}$$

$$= \frac{-1\pm\sqrt{1-8}}{4}$$

$$= \frac{-1\pm\sqrt{-7}}{4}$$

$x = \frac{-1\pm i\sqrt{7}}{4}$ which is a imaginary roots

$\therefore$ The rational root of the given equation is $x=1$

(ii) Given:

$x^8-3x+1=0$

$\therefore a_n = 1, \quad a_0 = 1$

If $\frac{p}{q}$ is a root of the Polynomial then $(p,q)=1$

p is a factor of $a_0 = 1$

q is a factor of $a_n = 1$

$\therefore p$ must divide 1 and q must divide 1

The Possible values of p are ± 1 and the Possible values of q are ± 1

Using these p and q we Can form only the fractions $\pm\frac{1}{1}$

Let $P(x) = x^8-3x+1$

Put $x=1$, $P(1) = (1)^8-3(1)+1$

$P(1) = 1-3+1 = -1 \neq 0$

$\therefore x = 1$ is not a root

Put $x=-1$, $P(-1) = (-1)^8-3(-1)+1$

$P(-1) = 1+3+1 = 5 \neq 0$

$\therefore x=-1$ is not a root

$\therefore$ The given equation has no rational roots

3) Solve: $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$

<u>Soln</u>: Given:

$$8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$$

$$8x^{\frac{3}{2n}} - \frac{8}{x^{\frac{3}{2n}}} = 63$$

Take $y = x^{\frac{3}{2n}}$

$8y - \frac{8}{y} = 63$

Multiple by y,

$8y^2 - 8 = 63y$ $\qquad -64$

$\wedge$

$8y^2-63y-8=0$ $\qquad -64 \quad 1$

$8y^2-64y+y-8=0$

$8y(y-8)+1(y-8)=0$

$(y-8)(8y+1)=0$

$y-8=0, \quad 8y+1=0$

$y=8, \quad y=\frac{-1}{8}$

Put $y = x^{\frac{3}{2n}}$

$x^{\frac{3}{2n}} = 8, \quad x^{\frac{3}{2n}} = \frac{-1}{8}$

$x^{\frac{3}{2n}} = 2^3, \quad x^{\frac{3}{2n}} = \left(\frac{-1}{2}\right)^3$

$$\left[x^{\frac{\cancel{3}}{2\cancel{n}}}\right]^{\frac{2\cancel{n}}{\cancel{3}}} = \left[2^{\cancel{3}}\right]^{\frac{2n}{\cancel{3}}} = (2^2)^n = 4^n$$

$\boxed{x = 4^n}$

and $\left[x^{\frac{\cancel{3}}{2\cancel{n}}}\right]^{\frac{2\cancel{n}}{\cancel{3}}} = \left[\left(\frac{-1}{2}\right)^{\cancel{3}}\right]^{\frac{2n}{\cancel{3}}}$

$$x = \left[\left(\frac{-1}{2}\right)^2\right]^n = \left(\frac{1}{4}\right)^n$$

$\boxed{x = \frac{1}{4^n}}$ which is not Possible in given eqn.

[Given: $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$

Now, put $x = 4^n$,

$8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}}$

$= 8(4^{\cancel{n}})^{\frac{3}{2\cancel{n}}} - 8(4^{\cancel{n}})^{-\frac{3}{2\cancel{n}}}$

$= 8(2^{\cancel{2}})^{\frac{3}{\cancel{2}}} - 8(2^{\cancel{2}})^{-\frac{3}{\cancel{2}}}$

$= 8\left[2^3 - 2^{-3}\right] = 8\left[2^3 - \frac{1}{2^3}\right]$

$= 8\left[8 - \frac{1}{8}\right] = 64 - 1 = 63$

Also, Put $x = \frac{1}{4^n}$

$8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}}$

$$= 8\left(\frac{1}{4^n}\right)^{\frac{3}{2n}} - 8\left(\frac{1}{4^n}\right)^{-\frac{3}{2n}}$$

$$= 8\left(\frac{1}{(4^{\cancel{n}})^{\frac{3}{2\cancel{n}}}}\right) - 8\left(\frac{1}{(4^{\cancel{n}})^{\frac{-3}{2\cancel{n}}}}\right)$$

$$= 8\left(\frac{1}{(2^{\cancel{2}})^{\frac{3}{\cancel{2}}}}\right) - 8\left(\frac{1}{(2^{\cancel{2}})^{\frac{-3}{\cancel{2}}}}\right)$$

$$= 8\left[\frac{1}{2^3} - \frac{1}{2^{-3}}\right]$$

$$= 8\left[\frac{1}{2^3} - 2^3\right]$$

$$= 8\left[\frac{1}{8} - 8\right]$$

$= 1 - 64 = -63$ which is not possible]

$\therefore x = 4^n$ is a root of the given equation

4) Solve: $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

<u>Soln</u>: Given:

$$2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$$

$$2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{1}{\frac{x}{a}}} = \frac{b}{a} + \frac{6a}{b}$$

$$2\sqrt{\frac{x}{a}} + 3\frac{1}{\sqrt{\frac{x}{a}}} = \frac{b}{a} + \frac{6a}{b}$$

Take $\sqrt{\frac{x}{a}} = y$

$$2y + 3\frac{1}{y} = \frac{b}{a} + \frac{6a}{b}$$

Multiple by y,

$$2y^2+3 = \frac{by}{a} + \frac{6ay}{b}$$

$$2y^2+3 = \frac{b^2y + 6a^2y}{ab}$$

$2aby^2 + 3ab = b^2y + 6a^2y$

$2aby^2 - b^2y - 6a^2y + 3ab = 0$

$by(2ay-b) - 3a(2ay-b) = 0$

$(by-3a)(2ay-b) = 0$

$by-3a=0, \quad 2ay-b=0$

$by = 3a, \quad 2ay = b$

$y = \frac{3a}{b}, \quad y = \frac{b}{2a}$

Put $y = \sqrt{\frac{x}{a}}$

$\sqrt{\frac{x}{a}} = \frac{3a}{b}, \quad \sqrt{\frac{x}{a}} = \frac{b}{2a}$

Squaring on both sides, we get,

$\frac{x}{a} = \frac{9a^2}{b^2}, \quad \frac{x}{\cancel{a}} = \frac{b^2}{4a^{\cancel{2}}}$

$x = \frac{9a^3}{b^2}, \quad x = \frac{b^2}{4a}$

$\therefore$ The Solution is $x = \frac{9a^3}{b^2}$ and $x = \frac{b^2}{4a}$

5) Solve the equations

(i) $6x^4-35x^3+62x^2-35x+6=0$

(ii) $x^4+3x^3-3x-1=0$

<u>Soln</u>:

Let $P(x)=x^9-5x^8-14x^7=0$

$+\ \ -\ \ -$

1

number of Sign Changes $=1$

∴ The maximum number of positive real roots is 1

Let $P(-x)=-x^9-5x^8+14x^7=0$

$-\ \ -\ \ +$

1

number of sign changes $=1$

∴ The maximum number of negative real roots is 1

It has atmost one Positive real root and atmost one negative real root

∴ Remaining 7 roots are zero.

Given: $x^9-5x^8-14x^7=0$

$x^7(x^2-5x-14)=0$

$x^7=0,\ x^2-5x-14=0$

$\boxed{x=0}$ (7 times)

$x^2-5x-14=0$

$(x-7)(x+2)=0$

$x-7=0,\ x+2=0$

$\boxed{x=7},\ \boxed{x=-2}$

5) Find the exact number of real roots and imaginary of the equation $x^9+9x^7+7x^5+5x^3+3x$

Soln: Let

$P(x)=x^9+9x^7+7x^5+5x^3+3x=0$

number of Sign Changes $=0$

∴ The maximum number of Positive real roots is 0

(ie) no Positive real roots

$P(-x)=-x^9-9x^7-7x^5-5x^3-3x=0$

number of Sign Changes $=0$

∴ The maximum number of negative real roots is 0

(ie) no negative real roots

But $x=0$ is a root

∴ It has no positive real roots and no negative real roots

Total number of roots $=9$

number of real roots $=1^{(-)}$

number of Imaginary roots $=8$

∴ number of real roots $=1$

number of Imaginary roots $=8$

EX 3.30

Show that the polynomial $9x^9+2x^5-x^4-7x^2+2$ has atleast six imaginary roots

Soln: Let

$P(x)=9x^9+2x^5-x^4-7x^2+2=0$

$+\ \ +\ \ -\ \ -\ \ +$

$1\ \ \ 2$

number of Sign Changes $=2$

∴ The maximum number of Positive real root is 2

Let

$P(-x)=-9x^9-2x^5-x^4-7x^2+2$

$-\ \ -\ \ -\ \ -\ \ +$

1

number of Sign changes $=1$

∴ The maximum number of negative real root is 1

But 0 is not a root

(ii) $x^2-4x+1=0$

$a=1,\ b=-4,\ c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$=\frac{4\pm\sqrt{16-4}}{2}$$

$$=\frac{4\pm\sqrt{12}}{2}$$

$$=\frac{4\pm2\sqrt{3}}{2}$$

$$=\frac{\cancel{2}(2\pm\sqrt{3})}{\cancel{2}}$$

$x=2\pm\sqrt{3}$

∴ The Solutions are $3+2\sqrt{2},\ 3-2\sqrt{2},\ 2+\sqrt{3},\ 2-\sqrt{3}$

EX 3.29 Find Solution, if any of the equation $2\cos^2x-9\cos x+4=0$

Soln: Given:

$2\cos^2x-9\cos x+4=0$

Take $\cos x=y$

$2y^2-9y+4=0$

$$\overset{8}{\wedge}$$
$-8\ \ -1$

$2y^2-8y-y+4=0$

$2y(y-4)-1(y-4)=0$

$(2y-1)(y-4)=0$

$2y-1=0,\ y-4=0$

$y=\frac{1}{2},\ y=4$

put $y=\cos x$

$\cos x=\frac{1}{2}$, $\cos x=4$ is not possible

∴ $\cos x=\frac{1}{2}\Rightarrow\cos x=\cos\frac{\pi}{3}$

∴ $\cos x=\frac{1}{2},\ x=2n\pi\pm\frac{\pi}{3},\ n\in Z$

∴ The given equation of the Soln is $x=2n\pi\pm\frac{\pi}{3},\ n\in Z$

3.9 Descartes Rule

EXERCISE 3.6

1) Discuss the maximum Possible numbers of Positive and negative roots of the Polynomial equation $9x^9-4x^8+4x^7-3x^6+2x^5+x^3+7x^2+7x+2=0$

Soln: Let

$P(x)=9x^9-4x^8+4x^7-3x^6+2x^5+x^3+7x^2+7x+2=0$

$+\ \ -\ \ +\ \ -\ \ +\ \ +\ \ +\ \ +\ \ +$

$1\ \ 2\ \ 3\ \ 4$

number of Sign Changes $=4$

∴ The maximum number of Positive real roots is 4

Let

$P(-x)=-9x^9-4x^8-4x^7-3x^6-2x^5-x^3+7x^2-7x+2=0$

$-\ \ -\ \ -\ \ -\ \ -\ \ -\ \ +\ \ -\ \ +$

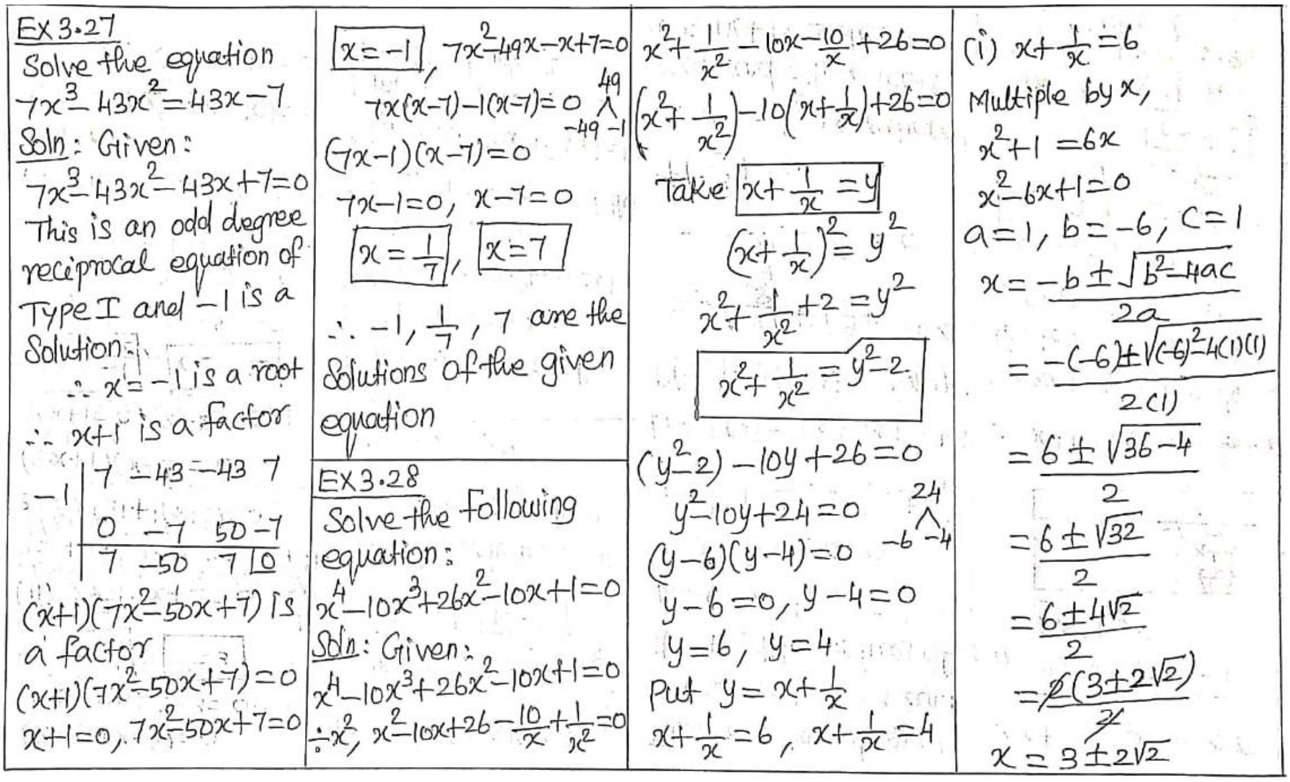
$1\ \ \ 2\ \ 3$

number of Sign Changes $=3$

∴ The maximum number of negative real roots is 3

It has atmost four Positive real roots and atmost three negative real roots

2) Discuss the maximum Possible number of Positive and negative roots of the Polynomial equations x^2-5x+6 and $x^2-5x+16$ Also draw rough sketch of the graphs.



EX 3.27

Solve the equation

$7x^3-43x^2=43x-7$

Soln: Given:

$7x^3-43x^2-43x+7=0$

This is an odd degree reciprocal equation of Type I and −1 is a Solution.

$\therefore x=-1$ is a root

$\therefore x+1$ is a factor

$$\begin{array}{r|rrrr} -1 & 7 & -43 & -43 & 7 \\ & 0 & -7 & 50 & -7 \\ \hline & 7 & -50 & 7 & \boxed{0} \end{array}$$

$(x+1)(7x^2-50x+7)$ is a factor

$(x+1)(7x^2-50x+7)=0$

$x+1=0, 7x^2-50x+7=0$

$\boxed{x=-1}$, $7x^2-49x-x+7=0$ $\overset{49}{\wedge}$ $-49\ -1$

$7x(x-7)-1(x-7)=0$

$(7x-1)(x-7)=0$

$7x-1=0, x-7=0$

$\boxed{x=\frac{1}{7}}$, $\boxed{x=7}$

$\therefore -1, \frac{1}{7}, 7$ are the Solutions of the given equation

EX 3.28

Solve the following equation:

$x^4-10x^3+26x^2-10x+1=0$

Soln: Given:

$x^4-10x^3+26x^2-10x+1=0$

$\div x^2$, $x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}=0$

$x^2+\frac{1}{x^2}-10x-\frac{10}{x}+26=0$

$\left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26=0$

Take $\boxed{x+\frac{1}{x}=y}$

$\left(x+\frac{1}{x}\right)^2=y^2$

$x^2+\frac{1}{x^2}+2=y^2$

$\boxed{x^2+\frac{1}{x^2}=y^2-2}$

$(y^2-2)-10y+26=0$

$y^2-10y+24=0$ $\overset{24}{\wedge}$ $-6\ -4$

$(y-6)(y-4)=0$

$y-6=0, y-4=0$

$y=6, y=4$

Put $y=x+\frac{1}{x}$

$x+\frac{1}{x}=6$, $x+\frac{1}{x}=4$

(i) $x+\frac{1}{x}=6$

Multiple by x,

$x^2+1=6x$

$x^2-6x+1=0$

$a=1, b=-6, c=1$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(1)}}{2(1)}$

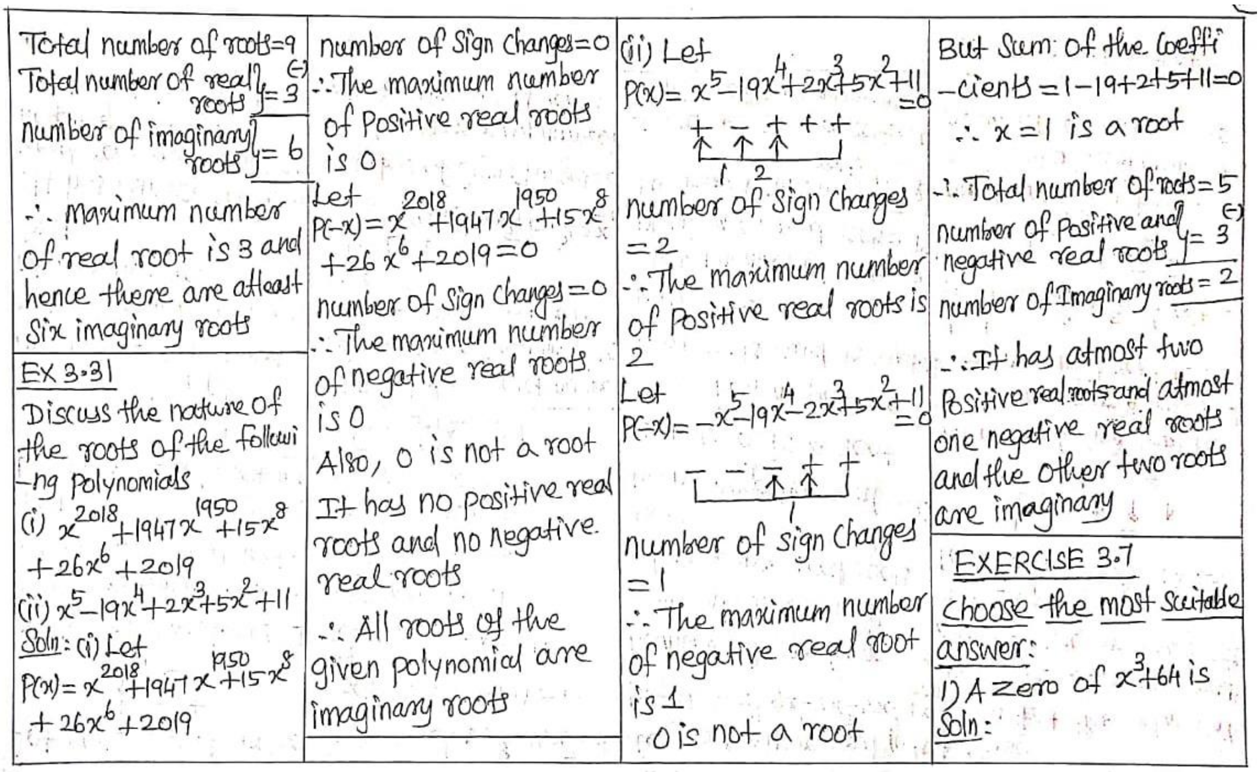
$=\frac{6\pm\sqrt{36-4}}{2}$

$=\frac{6\pm\sqrt{32}}{2}$

$=\frac{6\pm4\sqrt{2}}{2}$

$=\frac{\cancel{2}(3\pm2\sqrt{2})}{\cancel{2}}$

$x=3\pm2\sqrt{2}$



Total number of roots $=9$

Total number of real roots $\}=3$ (−)

number of imaginary roots $\}=6$

$\therefore$ Maximum number of real root is 3 and hence there are atleast Six imaginary roots

EX 3.31

Discuss the nature of the roots of the following polynomials

(i) $x^{2018}+1947x^{1950}+15x^8+26x^6+2019$

(ii) $x^5-19x^4+2x^3+5x^2+11$

Soln: (i) Let

$P(x)=x^{2018}+1947x^{1950}+15x^8+26x^6+2019$

number of Sign Changes $=0$

$\therefore$ The maximum number of Positive real roots is 0

Let

$P(-x)=x^{2018}+1947x^{1950}+15x^8+26x^6+2019=0$

number of Sign Changes $=0$

$\therefore$ The maximum number of negative real roots is 0

Also, 0 is not a root

It has no positive real roots and no negative real roots

$\therefore$ All roots of the given polynomial are imaginary roots

(ii) Let

$P(x)=x^5-19x^4+2x^3+5x^2+11=0$

$+\ -\ +\ +\ +$

1 2

number of Sign changes $=2$

$\therefore$ The maximum number of Positive real roots is 2

Let

$P(-x)=-x^5-19x^4-2x^3+5x^2+11=0$

$-\ -\ -\ +\ +$

1

number of sign Changes $=1$

$\therefore$ The maximum number of negative real root is 1

0 is not a root

But Sum of the Coefficients $=1-19+2+5+11=0$

$\therefore x=1$ is a root

$\therefore$ Total number of roots $=5$

number of Positive and negative real roots $\}=3$ (−)

number of Imaginary roots $=2$

$\therefore$ It has atmost two Positive real roots and atmost one negative real roots and the other two roots are imaginary

EXERCISE 3.7

Choose the most suitable answer:

1) A zero of x^3+64 is

Soln:

Ex 3.25

Solve the equation

$x^3-5x^2-4x+20=0$

Soln:

Let $P(x)=x^3-5x^2-4x+20$

$P(2)=(2)^3-5(2)^2-4(2)+20$

$=8-20-8+20=0$

$P(2)=0$

$\therefore x=2$ is a root

$\therefore x-2$ is a factor

$$\begin{array}{c|cccc} 2 & 1 & -5 & -4 & 20 \\ & 0 & 2 & -6 & -20 \\ \hline & 1 & -3 & -10 & \boxed{0} \end{array}$$

$(x-2)(x^2-3x-10)$ is a factor

$\therefore (x-2)(x^2-3x-10)=0$

$x-2=0,\ x^2-3x-10=0$

$\boxed{x=2},\ (x-5)(x+2)=0$ $\quad \overset{-10}{\wedge}\ -5\ 2$

$x-5=0,\ x+2=0$

$\boxed{x=5},\ \boxed{x=-2}$

$\therefore$ The Solutions of the given equation are $2, -2, 5$

Ex 3.26

Find the roots of $2x^3+3x^2+2x+3$

Soln: Given: $2x^3+3x^2+2x+3$

$\therefore a_n=2,\ a_0=3$

If $\frac{p}{q}$ is a root of the Polynomial then $(p,q)=1$

p must divide 3 and q must divide 2

The Possible values of p are $1, -1, 3, -3$ and the Possible values of q are $1, -1, 2, -2$

$\therefore \frac{p}{q}$ form are

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{1}$

Let $P(x)=2x^3+3x^2+2x+3$

$P\left(-\frac{3}{2}\right)=2\left(-\frac{3}{2}\right)^3+3\left(-\frac{3}{2}\right)^2+\cancel{2}\left(-\frac{3}{\cancel{2}}\right)+3$

$=\cancel{2}\left(-\frac{27}{\cancel{8}_4}\right)+3\left(\frac{9}{4}\right)\cancel{-3}\cancel{+3}$

$=-\frac{\cancel{27}}{\cancel{4}}+\frac{\cancel{27}}{\cancel{4}}=0$

$P\left(-\frac{3}{2}\right)=0$

$\therefore x=-\frac{3}{2}$ is rational root and other roots are not possible.

$2x+3$ is a factor

$$\begin{array}{r|l} & x^2+1 \\ \hline 2x+3 & \cancel{2x^3}+\cancel{3x^2}+2x+3 \\ & \cancel{2x^3}+\cancel{3x^2} \\ & \underline{-\qquad -} \\ \hline & \qquad \cancel{2x}+\cancel{3} \\ & \qquad \cancel{2x}+\cancel{3} \\ & \qquad \underline{-\quad -} \\ \hline & \qquad 0 \end{array}$$

$\therefore x^2+1$ is also a factor

$\therefore (2x+3)(x^2+1)$ is also a factor

$\therefore (2x+3)(x^2+1)=0$

$2x+3=0,\ x^2+1=0$

$\boxed{x=-\frac{3}{2}},\ x^2=-1$

$x=\sqrt{-1}$

$\boxed{x=\pm i}$

$\therefore$ The roots of the given equation are $-\frac{3}{2}$, i and $-i$

Given:

$6x^4-35x^3+62x^2-35x+6=0$

$\div x^2,\ 6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$

$6x^2+\frac{6}{x^2}-35x-\frac{35}{x}+62=0$

$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$

Take $\boxed{x+\frac{1}{x}=y} \Rightarrow \left(x+\frac{1}{x}\right)^2=y^2$

$x^2+\frac{1}{x^2}+2=y^2$

$\boxed{x^2+\frac{1}{x^2}=y^2-2}$

$6(y^2-2)-35y+62=0$

$6y^2-12-35y+62=0$

$6y^2-35y+50=0$ $\quad \overset{300}{\wedge}\ -15\ -20$

$6y^2-15y-20y+50=0$

$3y(2y-5)-10(2y-5)=0$

$(3y-10)(2y-5)=0$

$3y-10=0,\ 2y-5=0$

$y=\frac{10}{3},\ y=\frac{5}{2}$

$$\begin{array}{r|l} 2 & 300 \\ \hline 2 & 150 \\ \hline 5 & 75 \\ \hline & 15 \end{array}$$

Put $y=x+\frac{1}{x}$

$x+\frac{1}{x}=\frac{10}{3}$ and $x+\frac{1}{x}=\frac{5}{2}$

(i) $x+\frac{1}{x}=\frac{10}{3}$

Multiple by x, $x^2+1=\frac{10}{3}x$

$3x^2+3=10x$ $\quad \overset{9}{\wedge}\ -9\ -1$

$3x^2-10x+3=0$

$3x^2-9x-x+3=0$

$3x(x-3)-1(x-3)=0$

$(3x-1)(x-3)=0$

$3x-1=0,\ x-3=0$

$\boxed{x=\frac{1}{3}},\ \boxed{x=3}$

(ii) $x+\frac{1}{x}=\frac{5}{2}$

Multiple by x, $x^2+1=\frac{5}{2}x$

$2x^2+2=5x$ $\quad \overset{4}{\wedge}\ -4\ -1$

$2x^2-5x+2=0$

$2x^2-4x-x+2=0$

$2x(x-2)-1(x-2)=0$

$(2x-1)(x-2)=0$

$2x-1=0,\ x-2=0$

$\boxed{x=\frac{1}{2}},\ \boxed{x=2}$

$\therefore$ The required Solutions of the given equation are $2, \frac{1}{2}, 3, \frac{1}{3}$

(ii) Given: $x^4+3x^3-3x-1=0$

Let $P(x)=x^4+3x^3-3x-1$

$P(1)=(1)^4+3(1)^3-3(1)-1$

$P(1)=1+3-3-1=0$

$\therefore x=1$ is a root

$\therefore x-1$ is a factor

Also, $P(-1)=(-1)^4+3(-1)^3-3(-1)-1$

$P(-1)=1-3+3-1=0$

$\therefore x=-1$ is a root

$\therefore x+1$ is a factor

$\therefore (x-1)(x+1)$ is also a factor

$\therefore x^2-1$ is also a factor

$$\begin{array}{r|l} & x^2+3x+1 \\ \hline x^2-1 & \cancel{x^4}+3x^3+0x^2-3x-1 \\ & \cancel{x^4} \qquad \mp x^2 \\ \hline & \cancel{3x^3}+x^2-\cancel{3x} \\ & \cancel{3x^3} \qquad \mp \cancel{3x} \\ \hline & \qquad \cancel{x^2} \quad \cancel{-1} \\ & \qquad \cancel{x^2} \quad \cancel{-1} \\ & \qquad - \quad + \\ \hline & \qquad 0 \end{array}$$

$\therefore x^2+3x+1$ is a factor

$(x^2-1)(x^2+3x+1)$ is a factor

$(x^2-1)(x^2+3x+1)=0$

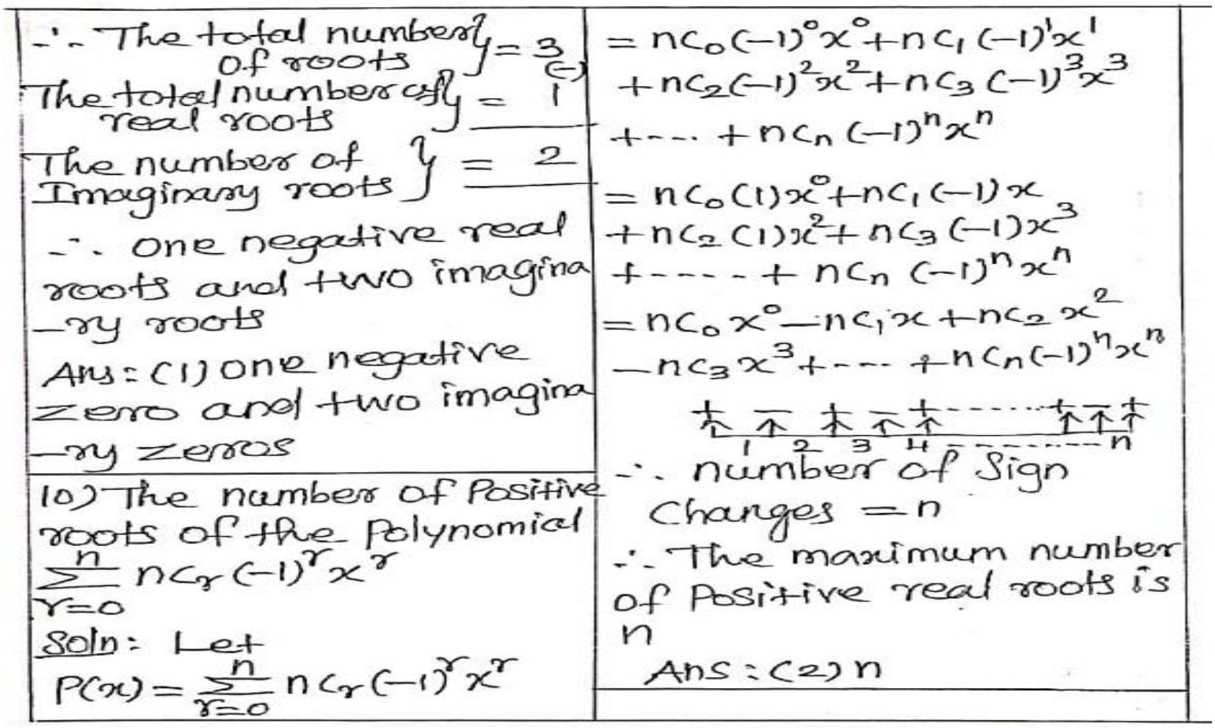
$x^2-1=0,\ x^2+3x+1=0$

$x^2=1 \qquad a=1, b=3, c=1$

$\boxed{x=\pm 1} \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$

$=\frac{-3\pm\sqrt{9-4}}{2}$



∴ The total number of roots } = 3

The total number of real roots } = 1

The number of Imaginary roots } = 2

∴ one negative real roots and two imaginary roots

Ans: (1) one negative zero and two imaginary zeros

10) The number of Positive roots of the Polynomial $\sum_{r=0}^{n} nC_r(-1)^r x^r$

Soln: Let

$P(x)=\sum_{r=0}^{n} nC_r(-1)^r x^r$

$= nC_0(-1)^0x^0+nC_1(-1)^1x^1+nC_2(-1)^2x^2+nC_3(-1)^3x^3+\cdots+nC_n(-1)^nx^n$

$= nC_0(1)x^0+nC_1(-1)x+nC_2(1)x^2+nC_3(-1)x^3+\cdots+nC_n(-1)^nx^n$

$= nC_0x^0-nC_1x+nC_2x^2-nC_3x^3+\cdots+nC_n(-1)^nx^n$

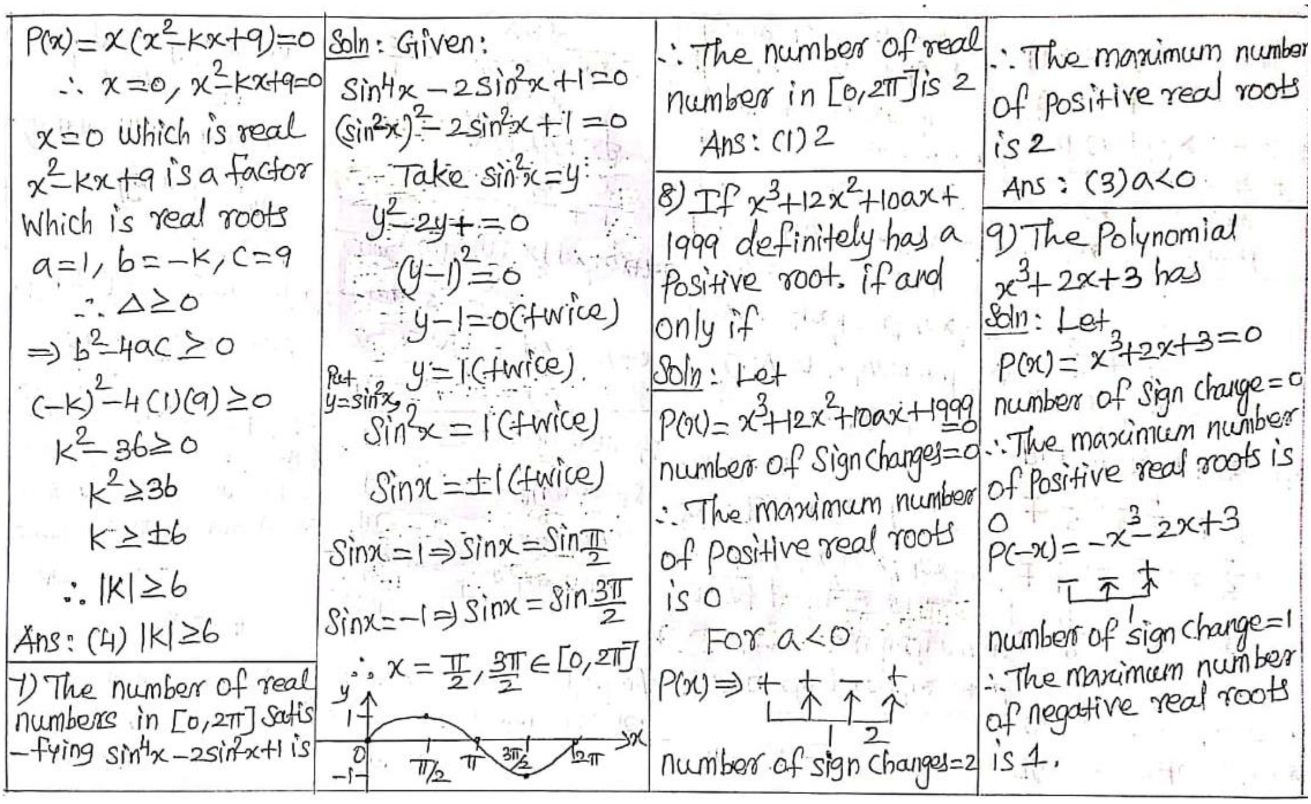
$+ \; - \; + \; - \; + \cdots + \; - \; +$

$1 \;\; 2 \;\; 3 \;\; 4 \cdots n$

∴ number of Sign Changes = n

∴ The maximum number of Positive real roots is n

Ans: (2) n



$P(x)=x(x^2-kx+9)=0$

∴ $x=0$, $x^2-kx+9=0$

$x=0$ which is real

x^2-kx+9 is a factor Which is real roots

$a=1, b=-k, c=9$

∴ $\Delta\geq 0$

$\Rightarrow b^2-4ac\geq 0$

$(-k)^2-4(1)(9)\geq 0$

$k^2-36\geq 0$

$k^2\geq 36$

$k\geq \pm 6$

∴ $|k|\geq 6$

Ans: (4) $|k|\geq 6$

7) The number of real numbers in $[0,2\pi]$ Satisfying $\sin^4x-2\sin^2x+1$ is

Soln: Given:

$\sin^4x-2\sin^2x+1=0$

$(\sin^2x)^2-2\sin^2x+1=0$

Take $\sin^2x=y$

$y^2-2y+=0$

$(y-1)^2=0$

$y-1=0$ (twice)

$y=1$ (twice)

Put $y=\sin^2x$,

$\sin^2x=1$ (twice)

$\sin x=\pm 1$ (twice)

$\sin x=1\Rightarrow \sin x=\sin\frac{\pi}{2}$

$\sin x=-1\Rightarrow \sin x=\sin\frac{3\pi}{2}$

∴ $x=\frac{\pi}{2},\frac{3\pi}{2}\in[0,2\pi]$

y
1
0
-1
$\pi/2$
π
$3\pi/2$
2π
x

∴ The number of real number in $[0,2\pi]$ is 2

Ans: (1) 2

8) If $x^3+12x^2+10ax+1999$ definitely has a positive root, if and only if

Soln: Let

$P(x)=x^3+12x^2+10ax+1999=0$

number of Sign changes = 0

∴ The maximum number of positive real roots is 0

For $a<0$

$P(x)\Rightarrow + \; + \; - \; +$

$1 \;\; 2$

number of sign changes = 2

∴ The maximum number of positive real roots is 2

Ans: (3) $a<0$

9) The Polynomial x^3+2x+3 has

Soln: Let

$P(x)=x^3+2x+3=0$

number of sign change = 0

∴ The maximum number of Positive real roots is 0

$P(-x)=-x^3-2x+3$

$- \; - \; +$

1

number of sign change = 1

∴ The maximum number of negative real roots is 1.

$\therefore$ n is odd and Even

Put $n = 2n$, $x = 2n\pi + (-1)^{2n}\frac{\pi}{2}$

$$x = 2n\pi + [(-1)^2]^n \frac{\pi}{2}$$

$$x = 2n\pi + (1)\frac{\pi}{2}$$

$$x = 2n\pi + \frac{\pi}{2},\ n \in Z$$

$\therefore$ The Solution of the given equation is $x = 2n\pi + \frac{\pi}{2}$, $n \in Z$

(ii) Given: $12x^3 + 8x = 29x^2 - 4$

$12x^3 - 29x^2 + 8x + 4 = 0$

Let $P(x) = 12x^3 - 29x^2 + 8x + 4$

Put $x = 2$,

$P(2) = 12(2)^3 - 29(2)^2 + 8(2) + 4$

$= 96 - 116 + 16 + 4 = 0$

$P(2) = 0$

$\therefore x = 2$ is a root

$\therefore x - 2$ is a factor

$$\begin{array}{c|cccc} & 12 & -29 & 8 & 4 \\ 2 & 0 & 24 & -10 & -4 \\ \hline & 12 & -5 & -2 & \boxed{0} \end{array}$$

$(x-2)(12x^2 - 5x - 2)$ is a factor.

$\therefore (x-2)(12x^2 - 5x - 2) = 0$

$x - 2 = 0$, $12x^2 - 5x - 2 = 0$

$\boxed{x = 2}$, $12x^2 - 8x + 3x - 2 = 0$ $\quad \overset{-24}{\underset{-8\ \ 3}{\wedge}}$

$4x(3x-2) + 1(3x-2) = 0$

$(4x+1)(3x-2) = 0$

$4x + 1 = 0$, $3x - 2 = 0$

$\boxed{x = \frac{-1}{4}}$, $\boxed{x = \frac{2}{3}}$

$\therefore 2, \frac{-1}{4}, \frac{2}{3}$ are the Solutions of the given equation.

2) Examine for the rational roots of

(i) $2x^3 - x^2 - 1 = 0$

(ii) $x^8 - 3x + 1 = 0$

Soln: Given: $2x^3 - x^2 - 1 = 0$

$\therefore a_n = 2$, $a_0 = -1$

If $\frac{p}{q}$ is a root of the Polynomial then $(p, q) = 1$

p is a factor of $a_0 = -1$

q is a factor of $a_n = 2$

p must divide -1 and q must divide 2

The possible values of p are ±1 and the possible values of q are ±2

Using these p and q we can form only the fractions $\pm\frac{1}{1}, \pm\frac{1}{2}$

Let $P(x) = 2x^3 - x^2 - 1$

Put $x = 1$, $P(1) = 2(1)^3 - (1)^2 - 1$

$P(1) = 2 - 1 - 1 = 0$

$\therefore x = 1$ is a root

Put $x = -1$, $P(-1) = 2(-1)^3 - (-1)^2 - 1$

$= 2(-1) - (1) - 1$

$P(-1) = -2 - 1 - 1 = -4 \neq 0$

$\therefore x = -1$ is not a root

Put $x = \frac{1}{2}$, $P(\frac{1}{2}) = 2(\frac{1}{2})^3 - (\frac{1}{2})^2 - 1$

$= \cancel{2}\left(\frac{1}{\cancel{8}_4}\right) - \frac{1}{4} - 1$

$P(\frac{1}{2}) = \frac{\cancel{1}}{\cancel{4}} - \frac{\cancel{1}}{\cancel{4}} - 1 = -1 \neq 0$

$\therefore x = \frac{1}{2}$ is not a root

Put $x = -\frac{1}{2}$, $P(-\frac{1}{2}) = 2(\frac{-1}{2})^3 - (\frac{-1}{2})^2 - 1$

$= \cancel{2}\left(\frac{-1}{\cancel{8}_4}\right) - \frac{1}{4} - 1$

$= \frac{-1-1-4}{4} = \frac{-\cancel{6}^3}{\cancel{4}_2}$

$P(-\frac{1}{2}) = -\frac{3}{2} \neq 0$

Soln:

Let $P(x) = x^2 - 5x + 6 = 0$

$+ \quad - \quad +$

$1 \quad 2$

number of sign changes = 2

$\therefore$ The maximum number of Positive real roots is 2

Let $P(-x) = x^2 + 5x + 6 = 0$

number of sign changes = 0

$\therefore$ The maximum number of negative real roots is 0

(ie) no negative real roots

P(x) has atmost two Positive real roots and no negative real roots

Let $Q(x) = x^2 - 5x + 16 = 0$

$+ \quad - \quad +$

$1 \quad 2$

number of sign changes = 2

$\therefore$ The maximum number of Positive real roots is 2

Let $Q(-x) = x^2 + 5x + 16 = 0$

number of sign changes = 0

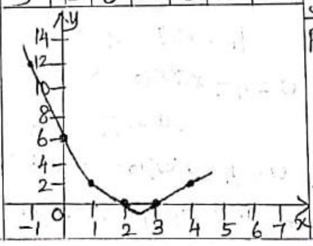
$\therefore$ The maximum number of negative real roots is 0

(ie) no negative real roots

$\therefore$ Q(x) has atmost two Positive real roots and no negative real roots

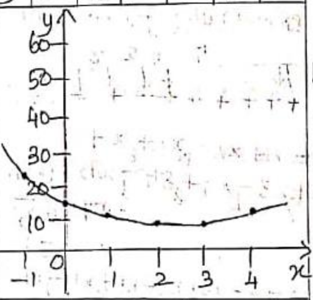
(i) $P(x) = x^2 - 5x + 6$

x	-1	0	1	2	3	4
y	12	6	2	0	0	2



(ii) $Q(x) = x^2 - 5x + 16$

x	-1	0	1	2	3	4
y	22	16	12	10	10	12



3) Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solutions

Soln: Let $P(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$

$+ \quad - \quad + \quad + \quad +$

$1 \quad 2$

number of sign changes = 2

$\therefore$ The maximum number of positive real roots is 2

Let $P(-x) = -x^9 + 5x^5 + 4x^4 + 2x^2 + ① = 0$

$- \quad + \quad + \quad + \quad +$

1

number of sign changes = 1

$\therefore$ The maximum number of negative real roots is 1

0 is not a root

Total number of roots = 9

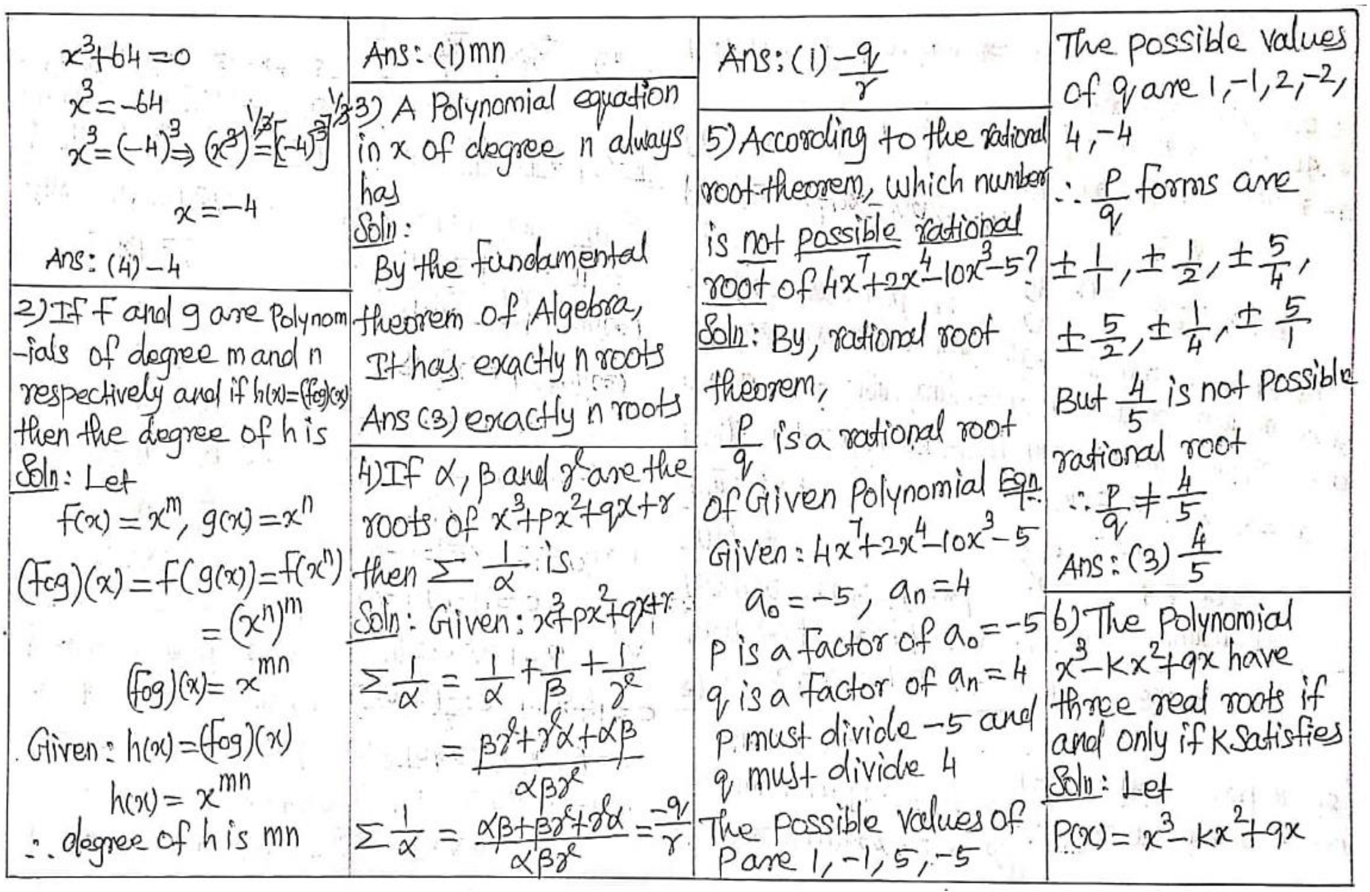
$\left.\begin{array}{l}\text{Maximum number of real roots} \\ [2+1]\end{array}\right\} = 3 \; (-)$

$\left.\begin{array}{l}\text{Minimum number of Imagi} \\ \text{-nary roots}\end{array}\right\} = 6$

$\therefore$ P(x) has atleast 6 imaginary roots

4) Determine the number of Positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$

Soln:



$x^3+64=0$

$x^3=-64$

$x^3=(-4)^3 \Rightarrow (x^3)^{1/3}=[(-4)^3]^{1/3}$

$x=-4$

Ans: (4) -4

2) If f and g are Polynomials of degree m and n respectively and if $h(x)=(f \circ g)(x)$ then the degree of h is

Soln: Let

$f(x)=x^m,\ g(x)=x^n$

$(f \circ g)(x)=f(g(x))=f(x^n)$

$=(x^n)^m$

$(f \circ g)(x)=x^{mn}$

Given: $h(x)=(f \circ g)(x)$

$h(x)=x^{mn}$

$\therefore$ degree of h is mn

Ans: (1) mn

3) A Polynomial equation in x of degree n always has

Soln:

By the fundamental theorem of Algebra, It has exactly n roots

Ans (3) exactly n roots

4) If α, β and γ are the roots of x^3+px^2+qx+r then $\Sigma \frac{1}{\alpha}$ is

Soln: Given: x^3+px^2+qx+r

$\Sigma \frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}$

$\Sigma \frac{1}{\alpha}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=\frac{-q}{r}$

Ans: (1) $\frac{-q}{r}$

5) According to the rational root theorem, which number is not possible rational root of $4x^7+2x^4-10x^3-5$?

Soln: By, rational root theorem,

$\frac{p}{q}$ is a rational root of Given Polynomial Eqn.

Given: $4x^7+2x^4-10x^3-5$

$a_0=-5,\ a_n=4$

p is a factor of $a_0=-5$

q is a factor of $a_n=4$

p must divide -5 and

q must divide 4

The possible values of P are $1, -1, 5, -5$

The possible values of q are $1, -1, 2, -2, 4, -4$

$\therefore \frac{p}{q}$ forms are

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{5}{4}, \pm\frac{5}{2}, \pm\frac{1}{4}, \pm\frac{5}{1}$

But $\frac{4}{5}$ is not Possible rational root

$\therefore \frac{p}{q} \neq \frac{4}{5}$

Ans: (3) $\frac{4}{5}$

6) The Polynomial x^3-kx^2+9x have three real roots if and only if k Satisfies

Soln: Let

$P(x)=x^3-kx^2+9x$