12. DISCRETE MATHEMATICS

EXERCISE 12.1.

1. Determine whether $*$ is a binary operation on the sets given below

(i) $a*b = a\cdot|b|$ on R.

Let $a, b \in R$, $|b| \in R$

$\therefore a|b| \in R$

$\Rightarrow a*b \in R \quad \forall a, b \in R$

$\therefore *$ is the binary operation on R.

(ii) $a*b = \min(a,b)$ on $A = \{1,2,3,4,5\}$

Let $a, b \in A$

$\min(a,b) \in A$

$a*b \in \quad \forall a, b \in A$

$*$ is the binary operation on A.

$\min(1,2) = 1 \in A$
$\min(3,5) = 3 \in A$
$\min(5,2) = 2 \in A$

(iii) $a*b = a\sqrt{b}$ is binary on R.

Let $a, b \in R$ but $\sqrt{(-ve)} \notin R$.

because the value of a and b are either positive or negative.

$\Rightarrow a\sqrt{b} \notin R$

$\therefore *$ is not binary on R.

2. On Z, define $\otimes$ by

$m \otimes n = m^n + n^m \quad \forall m, n \in Z$.

IS $\otimes$ is binary on Z.

$\rightarrow$ Let $m, n \in Z$.

take $m = 2, n = -2$.

$m^n + n^m = (2)^{-2} + (-2)^2$

$= \frac{1}{2^2} + 4 = \frac{1}{4} + 4 = \frac{17}{4} \notin Z$

$\Rightarrow m^n + n^m \notin Z. \Rightarrow m \otimes n \notin Z$.

$\therefore \otimes$ is not a binary operation

4. Let $A = \{a + \sqrt{5}b,\ a, b \in Z\}$ check whether the usual multiplication is a binary operation on A.

Let $x = a + \sqrt{5}b$, $y = c + \sqrt{5}d$.

$x, y \in A$ & $a, b, c, d \in Z$.

$xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$

$= ac + \sqrt{5}ad + \sqrt{5}bc + 5bd$.

$xy = (ac + 5bd) + \sqrt{5}(ad + bc) \in A$.

$\therefore xy \in A$

multiplication is binary on A.

3. Let $*$ be defined on R by $a*b = a + b + ab - 7$. IS $*$ binary on R? If so find $3*\left(\frac{-7}{15}\right)$.

$\rightarrow$ Let $a, b \in R$

clearly, $a, b, ab \in R$

$\therefore a + b + ab - 7 \in R$

$\Rightarrow a*b \in R$.

$*$ is binary operation on R.

Now,

$3*\left(-\frac{7}{15}\right) = 3 - \frac{7}{15} + 3\left(\frac{-7}{15}\right) - 7$

$= \frac{45 - 7 - 21 - 105}{15}$

$= \frac{45 - 133}{15}$

$3*\left(-\frac{7}{15}\right) = -\frac{88}{15}$.

5. (i) Define an operation $*$ on Q as follows

$a*b = \frac{a+b}{2}$; $a, b \in Q$

Examine the closure, commutative, associative properties satisfied by $*$ on Q.

$\rightarrow$

Closure axiom:

Let $a, b \in Q$

$\Rightarrow a + b \in Q$

$\Rightarrow \frac{a+b}{2} \in Q$

$\Rightarrow a*b \in Q \quad \forall a, b \in Q$.

$*$ is closure on Q.

Associative axiom:

Let $a, b, c \in Q \Rightarrow a*(b*c)$

$= (a*b)*c$.

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$a*(b*c) = a*\left(\frac{b+c}{2}\right)$

$= \frac{a+\left(\frac{b+c}{2}\right)}{2}$

$= \frac{2a+b+c}{4} \longrightarrow (1)$

$(a*b)*c = \left(\frac{a+b}{2}\right)*c$

$= \frac{\left(\frac{a+b}{2}\right)+c}{2}$

$= \frac{a+b+2c}{4}. \longrightarrow (2)$

From (1) and (2)

$a*(b*c) \neq (a*b)*c.$

$*$ is not associative on Q.

Commutative property:

Let $a, b \in Q$

$a*b = \frac{a+b}{2}$

$= \frac{b+a}{2}$

$a*b = b*a$

$*$ is commutative on Q.

(ii) Determine an operation $*$ on Q as follows $a*b = \frac{a+b}{2}$, $a,b \in Q$. Examine the existence of identity and existence of inverse for the operation $*$ on Q.

1. existence of identity:

Let $a \in Q$

'e' be the identity element on Q.

by definition of $*$, $a*e = \frac{a+e}{2}$.

by definition of e, $a*e = a$

$\Rightarrow \frac{a+e}{2} = a$

$a+e = 2a$

$e = 2a-a$

$e = a, \ \forall a \in Q.$

This means every element is a identity element.

(ii) existence of inverse.

$*$ has no identity element

$\therefore$ we cannot defined as

$a*a^{-1} = a^{-1}*a = e.$

$\therefore$ $*$ has no inverse.

6. Fill in the following table so that the binary operation $*$ on $A = \{a,b,c\}$ is commutative.

*	a	b	c
a	b		
b	c	b	a
c	a		c

$\rightarrow$ $A = \{a,b,c\}$

$*$ is commutative.

Required table.

$b*a = c \Rightarrow a*b = c$

$c*a = a \Rightarrow a*c = a$

$b*c = a \Rightarrow c*b = a$

*	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7. Consider the binary operation $*$ defined on the set $A = \{a,b,c,d\}$ by the following table,

*	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

(i) Commutative:

$a*b = c$ but $b*a = d \Rightarrow a*b \neq b*a.$

$a*c = b$ but $c*a = c$

$\Rightarrow a*c \neq c*a.$

(iii) Associative:

$(a*b)*c = c*c = a$

$a*(b*c) = a*b = c$.

$\therefore (a*b)*c \neq a*(b*c)$.

$*$ is not associative.

8. Let $A=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix}$

$B=\begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&1\end{pmatrix}$, $C=\begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$

be any three boolean matrices of the same type.

Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

$\rightarrow$

(i) $A \vee B = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&0&0&1\end{pmatrix}$

(ii) $A \wedge B = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$

(iii) $(A \vee B) \wedge C = \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&0&0&1\end{pmatrix}$

(iv) $(A \wedge B) \vee C = \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$

9. (i) Let $M = \left\{ \begin{pmatrix}x&x\\x&x\end{pmatrix} : x \in R-\{0\} \right\}$ and let $*$ be the matrices multiplication. Determine whether M is closed under $*$, If so examine the commutative and associative properties satisfied by $*$ on M.

$\rightarrow$

$M = \left\{ \begin{pmatrix}x&x\\x&x\end{pmatrix} : x \in R-\{0\} \right\}$

closure:

Let $A = \begin{pmatrix}x&x\\x&x\end{pmatrix} \in M$

$B = \begin{pmatrix}y&y\\y&y\end{pmatrix} \in M$.

$A*B = \begin{pmatrix}2xy&2xy\\2xy&2xy\end{pmatrix} \in M$.

Since $x \neq 0, y \neq 0 \Rightarrow 2xy \neq 0$

(i.e) $A*B \in M$.

closure axiom is true.

Commutative:

$A*B = \begin{pmatrix}2xy&2xy\\2xy&2xy\end{pmatrix}$

$= \begin{pmatrix}2yx&2yx\\2yx&2yx\end{pmatrix}$

$= \begin{pmatrix}y&y\\y&y\end{pmatrix}\begin{pmatrix}x&x\\x&x\end{pmatrix}$

$A*B = B*A$

$*$ is commutative on M.

Associative:

Matrix multiplication is always associative.

(i.e) $A*(B*C) = (A*B)*C$, $\forall A, B, C \in M$

$*$ is associative on M.

(ii) Let $M = \left\{ \begin{pmatrix}x&x\\x&x\end{pmatrix} ; x \in R-\{0\} \right\}$ and $*$ be the matrix multiplication. Determine whether $*$ is closed under $*$. If so examine the existence of identity, inverse property for $*$ on M.

$\rightarrow$

closure: 9 (i)

identity:

Let $E = \begin{pmatrix}e&e\\e&e\end{pmatrix}$ be the identity matrix.

by def, $AE = A$

(3)

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$\Rightarrow 2xe = x$

$e = \frac{x}{2x}$

$e = \frac{1}{2} \in R - \{0\}$.

$\therefore E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in M$.

inverse:

Let $A^{-1} = \begin{bmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{bmatrix}$ be the inverse of $A \in M$.

by def, $AA^{-1} = E$.

$$\begin{bmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$\Rightarrow 2xx^{-1} = \frac{1}{2}$

$x^{-1} = \frac{1}{2} \times \frac{1}{2x}$

$x^{-1} = \frac{1}{4x} \in R - \{0\}$.

$\therefore A^{-1} = \begin{bmatrix} 1/4x & 1/4x \\ 1/4x & 1/4x \end{bmatrix} \in M$.

10.(i) Let A be $Q - \{1\}$, Define $*$ on A by $x*y = x+y-xy$. Is $*$ is binary on A? If so example the commutative and associative properties satisfied by $*$ on A.

$\rightarrow$ $A = \{Q - \{1\}\}$

$x*y = x+y-xy$.

closure:

Let $x, y \in A \Rightarrow x \neq 1, y \neq 1$.

Since x and y are rational numbers.

$x+y-xy$ is also rational number.

To prove, $x*y \neq 1$.

Assume $x*y = 1$.

$x+y-xy = 1$

$y - xy = 1 - x$

$y(1-x) = 1-x$

$y = 1$.

but $y \neq 1$.

$\Rightarrow y = 1$ contradict $y \neq 1$

So our assumption $x*y = 1$ is wrong.

$\Rightarrow x*y \neq 1$.

$\therefore x, y \in A \Rightarrow x*y \in A$.

$*$ is closed on A.

Commutative:

Let $x, y \in A$

$x*y = x+y-xy$

$= y+x-yx$

$x*y = y*x$

$\therefore (A, *)$ is commutative.

Associative:

Let $x, y, z \in A$.

$x*(y*z) = x*(y+z-yz)$

$= x+(y+z-yz) - x(y+z-yz)$

$= x+y+z-yz-xy-xz+xyz$

$x*(y*z) = x+y+z-xy-yz-zx+xyz$. $\rightarrow ①$

$(x*y)*z = (x+y-xy)*z$

$= x+y-xy+z-(x+y-xy)z$

$(x*y)*z = x+y+z-xy-yz-zx+xyz$. $\rightarrow ②$

From ① and ②

$x*(y*z) = (x*y)*z$.

$(A, *)$ is associative. ④

(ii) Examine the existence of identity, inverse properties for $*$ on A. $A = \{ \mathbb{Q} - \{1\} \}$

identity:

$x * y = x + y - xy$

Let $x \in A$

Let 'e' be the identity element

by def, $x * e = x$

$x + e - xe = x$

$e(1-x) = x - x$

$e(1-x) = 0$

$e = 0$ | $1 - x = 0$ is not applicable.

$e = 0 \in A$.

$*$ has identity element on A.

inverse:

Let x^{-1} be the inverse of $x \in A$.

by def, $x * x^{-1} = e$

$x + x^{-1} - x x^{-1} = 0$

$x^{-1}(1-x) = -x$

$x^{-1} = -\frac{x}{1-x} \in A$. (or) $x^{-1} = \frac{x}{x-1} \in A$.

$*$ has inverse element $\forall x \in A$.

EXAMPLE PROBLEMS

Example 12.1.

Examine the binary operation (closure property) of the following operations on the respective sets (if it is not, make it binary).

(i) $a * b = a + 3ab - 5b^2, \forall a, b \in \mathbb{Z}$

$\longrightarrow$

Let $a, b \in \mathbb{Z}$

Then $3ab, 5b^2 \in \mathbb{Z}$

$\therefore a + 3ab - 5b^2 \in \mathbb{Z}$

$\Rightarrow a * b \in \mathbb{Z}$.

$\therefore$ closure property is true.

$\Rightarrow$ $*$ is a binary operation on $\mathbb{Z}$.

(ii) $a * b = \dfrac{a-b}{b-1}, \forall a, b \in \mathbb{Q}$.

Given,

$a * b$ is in the quotient form.

Since the division by 0 is undefined.

So, $b - 1 \neq 0 \Rightarrow b \neq 1$.

It is clear that,

$b - 1 = 0$ if $b = 1$. As $1 \in \mathbb{Q}$.

$*$ is not an binary operation on $\mathbb{Q}$.

But omitting 1 from $\mathbb{Q}$

$\therefore a * b$ exists in $\mathbb{Q} \setminus \{1\}$.

$\Rightarrow$ $*$ is a binary operation on $\mathbb{Q} - \{1\}$.

Example 12.2.

Verify the (i) closure property

(ii) Commutative property

(iii) Associative property

(iv) existense of identity.

(v) existense of inverse

for the arithmetic operation $+$ on $\mathbb{Z}$.

$\rightarrow$

(i) closure property:

Let $m, n \in \mathbb{Z}$.

Then $m + n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$.

closure property is true.

Hence '+' is a binary operation on $\mathbb{Z}$.

(ii) commutative property.

Let $m, n \in \mathbb{Z}$.

Then $m + n = n + m \quad \forall m, n \in \mathbb{Z}$

Commutative property is true.

(iii) Associative:

Let $m, n, p \in \mathbb{Z}$.

Then $(m+n) + p = m + (n+p) \quad \forall m, n, p \in \mathbb{Z}$.

$\therefore$ Associative property is true.

(iv) Existense of identity:

Let $e \in \mathbb{Z}$ be the identity element.

Then, $m + e = e + m = m$

$\Rightarrow e = 0 \in \mathbb{Z}. \quad \forall m \in \mathbb{Z}$.

(i.e) $m + 0 = 0 + m = m$.

identity property is true.

(v) Existense of inverse:

Let m' be the additive inverse of m.

Then, $m + m' = m' + m = 0$

$\Rightarrow m' = -m$.

(i.e) $m + (-m) = (-m) + m = 0$

inverse property assured.

'+' on $\mathbb{Z}$ Satisfies all the above five properties.

Example: 12.3.

Verify the (i) closure (ii) commutative (iii) associative (iv) existence of identity. (v) existence of inverse for the arithmetic operation $-$ on Z.

$\longrightarrow$

(i) closure:

Though '$-$' is not binary on N

Let $m, n \in Z$.

Then $m-n \in Z, \forall m, n \in Z$.

$\therefore$ closure property is true.

Hence,

'$-$' is binary on Z.

(ii) Commutative:

Take $m = 4 \in Z$, $n = 5 \in Z$.

then $m-n = 4-5 = -1 \in Z$

$n-m = 5-4 = 1 \in Z$.

$\Rightarrow m-n \neq n-m$.

The operation '$-$' is not commutative on Z

Commutative property is not true.

(iii) Associative:

Let $m=4, n=5$ and $p=7$

Then, $(m-n)-p = (4-5)-7 = -1-7 = -8 \in Z$

$m-(n-p) = 4-(5-7) = 4-(-2)$
$= 4+2$
$= 6, \in Z$

$\Rightarrow (m-n)-p \neq m-(n-p)$

$\therefore$ The operation '$-$' is not Associative.

(iv) existence of identity

identity doesnot exists.

(v) existence of inverse:

Inverse doesnot exists.

Example 12.4.

Verify (i) closure property, (ii) commutative property (iii) associative property (iv) existence of identity (v) existence of inverse for the arithmetic operation $+$ on Z_e = the set of all even integers.

$\longrightarrow$

set of all even integers.

$Z_e = \{2k \mid k \in Z\}$

$= \{\dots -6, -4, -2, 0, 2, 4 \dots\}$

(i) closure:

The sum of any two even integers is also an even integer.

Let $x, y \in Z_e$

Then $x = 2m, y = 2n \ \forall m, n \in Z$

$x+y = 2m+2n$

$= 2(m+n) \in Z_e$

closure axiom is true.

$\therefore$ '$+$' is a binary operation on Z_e.

(ii) Commutative:

$\forall x, y \in Z$

$x = 2m, y = 2n \ \forall m, n \in Z$.

$x+y = 2(m+n)$
$= 2(n+m)$
$= 2n+2m$

$x+y = y+x$

$\therefore$ + has the Commutative property on Z_e.

(iii) Associative:

Let $x, y, z \in Z_e$

Then $x = 2m$, $y = 2n$, $z = 2p$ $\forall m,n,p \in Z$

$(x+y)+z = (2m+2n)+2p$

$= 2m+(2n+2p)$

$(x+y)+z = x+(y+z)$

In general, Addition is always Associative.

(iv) Existence of identity.

Let 'e' be the identity element

$\therefore e = 0 \in Z_e$.

Let $x = 2k$, $\forall x \in Z_e$.

$x+e = e+x = x \Rightarrow e = 0$

$2k+e = e+2k = 2k \Rightarrow e = 0$.

$e = 0$ is the identity element

Moreover, Additive identity is 0.

identity property is true.

(v) Existence of inverse:

Let $x = 2k$, $x' = -2k$ $\forall x, x' \in Z_e$

Then,

$x+(-x) = -x+x = e$

$2k+(-2k) = (-2k)+2k = 0$

$\therefore -x$ is the inverse of $x \in Z_e$.

Inverse property is true.

Example 12.5.

Verify (i) closure (ii) Commutative (iii) associative (iv) existence of identity (v) existence of inverse for the arithmetic operation + on Z_o – the set of all odd integers.

$\rightarrow$

Z_o = { the set of all odd integers }

$= \{2k+1 : k \in Z\}$

$= \{ \ldots\ldots -5, -3, -1, 1, 3, 5, \ldots \}$

closure: the sum of any two odd integers is not an odd integer (even integer).

Let $x = 2m+1$,

$y = 2n+1$

$x+y = 2m+1 + 2n+1$.

$= 2(m+n)+2$. is even $\forall m,n \in Z$.

In general,

if $x, y \in Z_o$, then $x+y \notin Z_o$.

$\therefore$ '+' is not a binary operation on Z_o.

Other properties need not be checked as it is not a binary operation.

Example 12.6.

Verify (i) closure (ii) Commutative and (iii) associative property of the following operation on the given set.

$a * b = a^b$; $\forall a, b \in N$.

(exponentiation property)

$\rightarrow$

(i) closure:

Let $a, b \in N$

then $a * b = a^b \in N$ $\forall a, b \in N$

$*$ is a binary operation on N.

$\therefore$ closure property is true.

(ii) Commutative:

$a * b = a^b$ & $b * a = b^a$

* Put $a = 2$, $b = 3$

$a * b = 2^3 = 8$ & $b * a = 3^2 = 9$.

* but taking $a = 2$, $b = 2$.

$a * b = 2^2 = 4$ & $b * a = 2^2 = 4$.

by def, $a * b \neq b * a$.

$*$ doesnot have commutative property.

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iii) Associative:

$a*(b*c) = a*b^c = a^{(b^c)}$

Take, $a=2,\ b=3,\ c=4.$

$2*(3*4) = 2*(3^4)$
$= 2*81$
$= 2^{81}.$

$(a*b)*c = a^b*c = (a^b)^c = a^{bc}$

$(2*3)*4 = 2^3*4$
$= (2^3)^4 = 2^{12}.$

Hence,
$a*(b*c) \neq (a*b)*c.$

'$*$' doesnot have associative property on N.

note: This binary operation has no identity and no inverse.

Example 12.7

verify i) closure ii) Commutative iii) Associative iv) existence of identity v) existence of inverse for the following operation on the given set.

$m*n = m+n-mn,\ m,n \in Z.$

i) closure:
Let $m,n \in Z.$
Then $m*n = m+n-mn \in Z.$
$*$ is binary on Z.
closure axiom is true.

ii) Commutative:
$m*n = m+n-mn$
$= n+m-nm$
$m*n = n*m \quad \forall\ m,n \in Z.$
$*$ has Commutative property.

iii) Associative:
Let $m,n,p \in Z.$
$m*(n*p) = m*(n+p-np)$
$= m+(n+p-np)-m(n+p-np)$
$= m+n+p-np-mn-mp+mnp. \to ①$

$(m*n)*p = (m+n-mn)*p$
$= (m+n-mn)+p-(m+n-mn)p$
$= m+n+p-mn-mp-np+mnp \to ②$

From ① and ②
$m*(n*p) = (m*n)*p.$
Hence $*$ has Associative Property.

iv) Identity:
Let $e \in Z$ be the identity element.
by def, $m*e = m$
$m+e-me = m$
$e(1-m) = m-m$
$e(1-m) = 0$
$e = 0$ (or) $1-m=0$
$-m=-1$
$m=1.$

But m is arbitrary integer and hence need not be equal to 1.

$\Rightarrow$ $m=1$ is not applicable.

$\therefore e = 0 \in Z$.

Also, $m * 0 = 0 * m = m$, $\forall m \in Z$

Hence the existence of identity is assured.

(v) Existence of inverse:

Let m' be the inverse of $m \in Z$.

by def, $m * m' = e$

$$m + m' - mm' = 0$$

$$m'(1-m) = -m$$

$$m' = -\frac{m}{1-m}$$

$$m' = \frac{m}{m-1}$$

When $m=1 \Rightarrow m'$ is not defined.

When $m=2 \Rightarrow m'$ is an integer

But except $m=2 \Rightarrow m'$ need not be an integer for all values of m.

Hence inverse doesnot exists in Z.

Example 12.8.

Let $A=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

$\rightarrow$ Given that,

$$A=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Example 12.9.

Verify (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity (v) existence of inverse for the operation $+_5$ on Z_5, using table corresponding to addition modulo 5.

$\rightarrow$ $Z_5 = \{[0],[1],[2],[3],[4]\}$ ← to represent the classes $\{[0],[1],[2],[3],[4]\}$

We take remainders $\{0,1,2,3,4\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) closure:

Since each box in the table is filled by exactly one element of Z_5, the output $a +_5 b$ is unique.

Hence $+_5$ is a binary on Z_5.

closure axiom is true.

(ii) Commutative:

The entries are symmetrically placed with respect to main diagonal.

So $+_5$ has commutative property.

(iii) Associative:

Addition modulo 5 is always associative.

From the table,

$$(2 +_5 3) +_5 4 = 0 +_5 4 = 4 \pmod 5)$$

$2 +_5 (3 +_5 4) = 2 +_5 2 = 4 \pmod 5$

Hence,

$(2 +_5 3) +_5 4 = 2 +_5 (3 +_5 4)$

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that $+_5$ is associative

(iv) Existence of identity:

The row headed by 0 and the column headed by 0 are identical.

Hence the identical element is 0.

(V) Existence of inverse:

From the table,

element	inverse
0	0
1	4
2	3
3	2
4	1

Example : 12.10.

Verify (i) closure axiom (ii) commutative property (iii) associative (iv) existence of identity (v) existence of inverse for the operation X_{11} on the subset $A = \{1, 3, 4, 5, 9\}$ of the sets of remainders $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

→

$A = \{1, 3, 4, 5, 9\}$

X_{11}	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

From the table,

(i) closure:

All the entries in the table are members of A.

∴ X_{11} is a binary on A.

closure property is true.

(ii) commutative:

The entries are symmetrical about the main diagonal.

Hence X_{11} has commutative property.

(iii) Associative:

multiplication modulo 11 is always associative.

∴ Associative property is true.

(iv) identity:

The entries of both the row and column headed by the element 1 are identical.

Hence, the identity element is 1.

∴ identity property is true.

(V) inverse:

element	inverse.
1	1
3	4
4	3
5	9
9	5.

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Theorem: 12.1.

Uniqueness of identity.

If an algebraic structure, the identity element (if exists) must be unique.

Proof:

Let $(S, *)$ be an algebraic structure.
Assume that the identity element of S exists in S.

It is to be proved that the identity element is unique.

Suppose that e_1 and e_2 be any two identity elements of S.

Treating 'e_1' as identity element

$$e_2 * e_1 = e_1 * e_2 = e_2 \rightarrow (1)$$

Treating 'e_2' as the identity element

$$e_2 * e_1 = e_1 * e_2 = e_1 \longrightarrow (2)$$

From (1) and (2)

$$e_1 = e_2.$$

Hence the identity element is unique.

Theorem: 12.2.

Uniqueness of inverse.

In an algebraic structure the inverse of an element (if exists) must be unique.

Proof:

Let $(S, *)$ be an algebraic structure. and $a \in S$.

Let a_1 and a_2 are two inverse elements of $a \in S$.

Treating 'a_1' as inverse element

$$a * a_1 = a_1 * a = e \longrightarrow (1)$$

Treating 'a_2' as inverse element

$$a * a_2 = a_2 * a = e \rightarrow (2)$$

Now,

$$\begin{aligned} a_1 &= a_1 * e \\ &= a_1 * (a * a_2) \\ &= (a_1 * a) * a_2 \\ &= e * a_2 \end{aligned}$$

$$a_1 = a_2 \quad [\text{using } (1) \,\&\, (2)]$$

Hence inverse element of an element is unique.